

# Convergent Star Products for Projective Limits of Hilbert Spaces

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## Abstract

Given a locally convex vector space with a topology induced by Hilbert seminorms and a continuous bilinear form on it we construct a topology on its symmetric algebra such that the usual star product of exponential type becomes continuous. Many properties of the resulting locally convex algebra are explained. We compare this approach to various other discussions of convergent star products in finite and infinite dimensions. We pay special attention to the case of a Hilbert space and to nuclear spaces.

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## 1 Introduction

The canonical commutation relations

$$QP - PQ = i\hbar$$

are the paradigm of quantum physics. They indicate the transition from formerly commutative algebras of observables in classical mechanics to now non-commutative algebras, those generated by the fundamental variables of position  $Q$  and momentum  $P$ . While this basic form of the commutation relations is entirely algebraic, the need of physics is to have some more analytic framework. Traditionally, one views  $Q$  and  $P$  as (necessarily unbounded) self-adjoint operators on a Hilbert space. Then the commutation relation becomes immediately much more touchy as one has to take care of domains. Ultimately, the reasonable way to handle these difficulties is to use the Schrödinger representation which leads to a strongly continuous representation of the Heisenberg group. This way, the commutation relations encode an integration problem, namely from the infinitesimal picture of a Lie algebra representation by unbounded operators to the global picture of a group representation by unitary operators.

While this is all well-understood, things become more interesting in infinite dimensions: here one still has canonical commutation relations now based on a symplectic (or better: Poisson) structure on a vector space  $V$ . Physically, infinite dimensions correspond to a field theory with infinitely many degrees of freedom instead of a mechanical system. Then, algebraically, the commutation relations can be realized as a star product for the symmetric algebra over this vector space, see the seminal paper [1] where the basic notions of deformation quantization have been introduced as well as e.g. [10, 22, 24] for introductions. However, beyond the algebraic questions one is again interested in an analytic context: it turns out that now things are much more involved. First, there is no longer an essentially unique way to represent the canonical commutation relations by operators, a classical result which can be stated in many ways. One way to approach this non-uniqueness is now to focus first on the algebraic part and discuss the whole representation theory of this quantum algebra. To make this possible one has to go beyond the symmetric algebra and incorporate suitable completions instead.

Based on a  $C^*$ -algebraic formulation there are (at least) two approaches available. The classical one is to take formal exponentials of the unbounded quantities and implement a  $C^*$ -norm for the algebra they generate, see [14]. More recently, an alternative was proposed by taking formal resolvents and the  $C^*$ -algebra they generate [4]. These two approaches can be formulated in arbitrary dimensions and are used extensively in quantum field theory.

Only for finite dimensions there is a third  $C^*$ -algebraic way based on (strict) deformation quantization in the framework of Rieffel [19], see also [8, 9] for some more recent development: here one constructs a rather large  $C^*$ -algebra by deforming the bounded continuous functions on the underlying symplectic vector space. The deformation is based on certain oscillatory integrals which is the reason that this approach, though extremely appealing and powerful, will be restricted to finite dimensions. Nevertheless, in such finite-dimensional situations one has even ways to go beyond the flat situation and include also much more non-trivial geometries of the underlying geometric system, see e.g. [2]. Unfortunately, none of those techniques carry over to infinite dimensions.

While the  $C^*$ -algebraic approaches are very successful in many aspects, some questions seem to be hard to answer within this framework: from a deformation quantization point of view it is not completely obvious in which sense these algebras provide deformations of their classical counterparts, see, however, [3]. Closely related is the question of how one can get back the analogs of the classically unbounded quantities like polynomials on the symplectic vector space: in the quantum case they can not be elements of any  $C^*$ -algebra and thus they have to be recovered in certain well-behaved representations as unbounded operators on the representation space. This raises the question whether they can acquire some intrinsic meaning, independent of a chosen representation. In particular, all the  $C^*$ -algebraic constructions completely ignore possible additional structures on the underlying vector space  $V$ , like e.g. a given topology. This seems both from the purely mathematical but also from the physical point of view rather unpleasant.

In [23] a first step was taken to overcome some of these difficulties: instead of considering a  $C^*$ -algebraic construction, the polynomials, modeled as the symmetric algebra, were kept and quantized by means of a star product directly. Now the additional feature is that a given locally convex topology on the underlying vector space  $V$  induces a specific locally convex topology on the symmetric algebra  $\mathcal{S}^\bullet(V)$  in such a way that the star product becomes *continuous*. Necessarily, there will be no non-trivial sub-multiplicative seminorms, making the whole locally convex algebra quite non-trivial. It was then shown that in the completion the star product is a convergent series in the deformation parameter  $\hbar$ . This construction has good functorial properties and works for every locally convex space  $V$  with continuous constant Poisson structure. The basic feature was that on a fixed symmetric power  $\mathcal{S}^k(V)$  the *projective* locally convex topology was chosen. In finite dimensions this construction reproduces earlier versions [16, 17] of convergence results for the particular case of the Weyl-Moyal star product.

In the present paper we want to adapt the construction of [23] to the more particular case of a projective limit of (pre-) Hilbert spaces, i.e. a locally convex space where the topology is determined by Hilbert seminorms coming from (not necessarily non-degenerate) positive inner products. The major difference is now that for each fixed symmetric power  $\mathcal{S}^k(V)$  we have another choice of the topology, namely the one by extending the inner products first and taking the corresponding Hilbert seminorms afterwards. In general, this is coarser than the projective one and thus yields a larger and hence more interesting completion. We then use a star product coming from an arbitrary continuous bilinear form on  $V$ , thereby allowing for various other orderings beside the usual Weyl symmetrization. We are able to determine many features of this new algebra hosting the canonical commutation relations in arbitrary dimensions, including the convergence of the star product and an explicit description of the completion as certain analytic functions on the topological dual.

The paper is organized as follows: in Section 2 we outline the construction of the star product and the relevant topology. Since the star product is the usual one of exponential type on a vector space

we can be brief here. The topological properties are discussed in some detail, in particular as they differ at certain points significantly from the previous work [23]. After the necessary but technical estimates this results in the construction of the locally convex algebra in Theorem 2.13.

Section 3 contains various properties of the star-product algebra. First we show that a continuous antilinear involution on  $V$  extends to a continuous  $*$ -involution on the algebra. Then we are able to characterize the topology by some very simple conditions in Theorem 3.5, a feature which is absent in the case of [23]. The discussion of equivalences between different star products becomes now more involved as not all continuous symmetric bilinear forms give rise to equivalences as that was the case in [23]. Now in Theorem 3.10 we have to add a Hilbert-Schmidt condition similar to the one of Dito in [6]. In Theorem 3.26 we are able to characterize the completed star-product algebra as certain analytic functions on the topological dual. This will later be used to make contact to the more particular situation considered in [6]. In Theorem 3.31 we show the existence of many positive linear functionals provided the Poisson tensor allows for a compatible positive bilinear form of Hilbert-Schmidt type. Since the algebra is (necessarily) not locally multiplicatively convex, we have no general entire calculus. However, we can show that for elements of degree one, i.e. vectors in  $V$ , the star-exponential series converges absolutely. This is no longer true for quadratic elements, i.e. elements in  $\mathcal{S}^2(V)$ . However, we are able to show that in all GNS representations with respect to continuous positive linear functionals all elements up to quadratic ones yield essentially self-adjoint operators in Theorem 3.40. Here our topology is used in an essential way. The statement can be seen as a representation-independent version of Nelson's theorem, as it holds for arbitrary such GNS representations.

Finally, Section 4 contains a discussion of two particular cases of interest: First, we consider the case that  $V$  is not just a projective limit of Hilbert spaces but a Hilbert space directly. In this case, Dito discussed formal star products of exponential type and their formal equivalence in [6]. We can show that his algebra of functions contains our algebra, where the star product converges nicely, as a subalgebra. In this sense, we extend Dito's results from the formal power series context to a convergent one. In fact, we show a rather strong continuity with respect to the deformation parameter in Theorem 4.1.

The second case is a nuclear space  $V$ . It is well-known that any (complete) nuclear space can be seen as a projective limit of Hilbert spaces, see e.g. [11, Cor. 21.2.2]. Not very surprisingly, we prove that in this case our construction coincides with the previous one of [23] as for nuclear spaces the two competing notions of topological tensor products we use coincide. This way we can transfer the abstract characterization of the topology to the case of nuclear spaces in [23], a result which was missing in that approach. The important benefit from the projective Hilbert space point of view is now that we can show the existence of sufficiently many continuous positive linear functionals: an element in the completed  $*$ -algebra is zero iff all continuous positive functionals on it vanish. It follows that the resulting  $*$ -algebra has a faithful  $*$ -representation on a pre-Hilbert space, i.e. it is  $*$ -semisimple in the sense of [20].

**Notation:** For a set  $X$  and  $k \in \mathbb{N}_0$  we define  $X^k$  as the set of all functions from  $\{1, \dots, k\}$  (or the empty set if  $k = 0$ ) to  $X$  and usually put the parameter in the index, i.e.  $\{1, \dots, k\} \ni i \mapsto f_i \in X$  for  $f \in X^k$ . Let  $V$  be a vector space and  $k \in \mathbb{N}_0$ , then we write  $\mathcal{T}_{\text{alg}}^k(V)$  for the space of degree  $k$ -tensors over  $V$  and  $\mathcal{T}_{\text{alg}}^\bullet(V) := \bigoplus_{k \in \mathbb{N}_0} \mathcal{T}_{\text{alg}}^k(V)$  for the vector space underlying the tensor algebra. For  $x \in V^k$  we define the projections on the tensors of degree  $k$  by  $\langle \cdot \rangle_k: \mathcal{T}_{\text{alg}}^\bullet(V) \rightarrow \mathcal{T}_{\text{alg}}^k(V)$ . Let  $\mathfrak{S}_k \subseteq \{1, \dots, k\}^k$  be the symmetric group of degree  $k$  (in the case  $k = 0$  this is  $\mathfrak{S}_0 = \{\text{id}_\emptyset\}$ ), then  $\mathfrak{S}_k$  acts linearly on  $\mathcal{T}_{\text{alg}}^k(V)$  from the right via  $(x_1 \otimes \dots \otimes x_k)^\sigma := x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)}$ . This allows us to define the symmetrisation operators  $\mathcal{S}^k: \mathcal{T}_{\text{alg}}^k(V) \rightarrow \mathcal{T}_{\text{alg}}^k(V)$  by  $X \mapsto \mathcal{S}^k(X) := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} X^\sigma$  and  $\mathcal{S}^\bullet: \mathcal{T}_{\text{alg}}^\bullet(V) \rightarrow \mathcal{T}_{\text{alg}}^\bullet(V)$  by  $X \mapsto \mathcal{S}^\bullet(X) := \sum_{k \in \mathbb{N}_0} \mathcal{S}^k(\langle X \rangle_k)$ . These are projectors on subspaces of  $\mathcal{T}_{\text{alg}}^k(V)$  and  $\mathcal{T}_{\text{alg}}^\bullet(V)$  which we will denote by  $\mathcal{S}_{\text{alg}}^k(V)$  and  $\mathcal{S}_{\text{alg}}^\bullet(V)$ .

We will always denote an algebra as a pair  $(V, \circ)$  of a vector space  $V$  and a multiplication  $\circ$ ,

because we will discuss different products on the same vector space.

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## 2 Construction of the Algebra

As we want to construct a similar algebra like in [23], but by using Hilbert tensor products instead of projective tensor products, we have to restrict our attention to locally convex spaces whose topology is given by Hilbert seminorms.

Let  $V$  be a locally convex space, then a positive Hermitian form on  $V$  is a sesquilinear Hermitian and positive semi-definite form  $\langle \cdot | \cdot \rangle_\alpha: V \times V \rightarrow \mathbb{C}$  (antilinear in the first, linear in the second argument). By  $\mathcal{I}_V$  we denote the set of all continuous positive Hermitian forms on  $V$  and we will distinguish different positive Hermitian forms by a lowercase greek subscript. Out of  $p, q \geq 0$  and  $\langle \cdot | \cdot \rangle_\alpha, \langle \cdot | \cdot \rangle_\beta \in \mathcal{I}_V$  we get a new continuous positive Hermitian form  $\langle \cdot | \cdot \rangle_{p\alpha+q\beta} := p\langle \cdot | \cdot \rangle_\alpha + q\langle \cdot | \cdot \rangle_\beta$ .

Every  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_V$  yields a continuous Hilbert seminorm on  $V$ , defined as  $\|v\|_\alpha := \sqrt{\langle v | v \rangle_\alpha}$  for all  $v \in V$ . The set of all continuous Hilbert seminorms on  $V$  will be denoted by  $\mathcal{P}_V$ . Note that  $\|\cdot\|_{p\alpha+q\beta} = (q\|\cdot\|_\alpha^2 + p\|\cdot\|_\beta^2)^{1/2}$  and that  $\mathcal{P}_V$  with the usual partial ordering of seminorms (i.e. by pointwise comparison) is an upwards directed poset and that there is a one-to-one correspondence between  $\mathcal{I}_V$  and  $\mathcal{P}_V$  due to the polarisation identity.

In the following we will always assume that  $V$  is a Hausdorff locally convex space whose topology is given by its continuous Hilbert seminorms (“hilbertisable” in the language of [11]), i.e. we assume that  $\mathcal{P}_V$  is cofinal in the upwards directed set of all continuous seminorms on  $V$ . Important examples of such spaces are (pre-) Hilbert spaces and nuclear spaces (see [11, Corollary 21.2.2]) and, in general, all projective limits of pre-Hilbert spaces in the category of locally convex spaces.

### 2.1 Extension of Hilbert Seminorms to the Tensor Algebra

Analogous to [23], we extend all Hilbert seminorms from  $V$  to  $\mathcal{T}_{\text{alg}}^\bullet(V)$  with the difference that we first extend the  $\langle \cdot | \cdot \rangle_\alpha$  and reconstruct the seminorms out of their extensions:

**Definition 2.1** *For every continuous positive Hermitian form  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_V$  we define the sesquilinear extension  $\langle \cdot | \cdot \rangle_\alpha^\bullet: \mathcal{T}_{\text{alg}}^\bullet(V) \times \mathcal{T}_{\text{alg}}^\bullet(V) \rightarrow \mathbb{C}$*

$$(X, Y) \mapsto \langle X | Y \rangle_\alpha^\bullet := \sum_{k=0}^{\infty} \langle \langle X \rangle_k | \langle Y \rangle_k \rangle_\alpha^\bullet, \quad (2.1)$$

where

$$\langle x_1 \otimes \cdots \otimes x_k | y_1 \otimes \cdots \otimes y_k \rangle_\alpha^\bullet := k! \prod_{m=1}^k \langle x_m | y_m \rangle_\alpha \quad (2.2)$$

for all  $k \in \mathbb{N}_0$  and all  $x, y \in V^k$ .

It is well-known that this is a positive Hermitian form on all homogeneous tensor spaces and then it is clear that  $\langle \cdot | \cdot \rangle_\alpha^\bullet$  is a positive Hermitian form on  $\mathcal{T}_{\text{alg}}^\bullet(V)$ . We write  $\|\cdot\|_\alpha^\bullet$  for the resulting seminorm on  $\mathcal{T}_{\text{alg}}^\bullet(V)$  and  $\mathcal{T}^\bullet(V)$  for the locally convex space of  $\mathcal{T}_{\text{alg}}^\bullet(V)$  with the topology defined by the extensions of all  $\|\cdot\|_\alpha \in \mathcal{P}_V$ . Analogously, we write  $\mathcal{T}^k(V)$ ,  $\mathcal{S}^k(V)$  and  $\mathcal{S}^\bullet(V)$  for the subspaces  $\mathcal{T}_{\text{alg}}^k(V)$ ,  $\mathcal{S}_{\text{alg}}^k(V)$  and  $\mathcal{S}_{\text{alg}}^\bullet(V)$  with the subspace topology. Note that  $\|\cdot\|_\alpha^\bullet \leq \|\cdot\|_\beta^\bullet$  holds if and only if  $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$ . Note that, in general, for a fixed tensor degree the resulting topology on  $\mathcal{T}^k(V)$  is *not* the projective topology used in [23].

The factor  $k!$  in (2.2) for the extensions of positive Hermitian forms corresponds to the factor  $(n!)^R$  for  $R = 1/2$  in [23, Eq. (3.7)] for the extensions of seminorms (where  $R = 1/2$  yields the coarsest topology for which the continuity of the star-product could be shown in [23]). We are only interested in this special case because of the characterization in Section 3.1.

The following is an easy consequence of the definition of the topology on  $\mathcal{T}^\bullet(V)$ :

**Proposition 2.2**  $\mathcal{T}^\bullet(V)$  is Hausdorff and is metrizable if and only if  $V$  is metrizable.

For working with these extensions of not necessarily positive definite positive Hermitian forms, the following technical lemma will be helpful:

**Lemma 2.3** Let  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_V$ ,  $k \in \mathbb{N}$  and  $X \in \mathcal{T}^k(V)$  be given. Then  $X$  can be expressed as  $X = X_0 + \tilde{X}$  with tensors  $X_0, \tilde{X} \in \mathcal{T}^k(V)$  that have the following properties:

- i.) One has  $\|X_0\|_\alpha^\bullet = 0$  and there exists a finite (possibly empty) set  $A$  and tuples  $x_a \in V^k$  for all  $a \in A$  that fulfill  $\prod_{n=1}^k \|x_{a,n}\|_\alpha = 0$  and  $X_0 = \sum_{a \in A} x_{a,1} \otimes \cdots \otimes x_{a,k}$ .
- ii.) There exist a  $d \in \mathbb{N}_0$  and a  $\langle \cdot | \cdot \rangle_\alpha$ -orthonormal tuple  $e \in V^d$  as well as complex coefficients  $X^{a'}$ , such that

$$\tilde{X} = \sum_{a' \in \{1, \dots, d\}^k} X^{a'} e_{a'_1} \otimes \cdots \otimes e_{a'_k} \quad \text{and} \quad \|X\|_\alpha^{\bullet,2} = \|\tilde{X}\|_\alpha^{\bullet,2} = k! \sum_{a' \in \{1, \dots, d\}^k} |X^{a'}|^2. \quad (2.3)$$

PROOF: We can express  $X$  as a finite sum of simple tensors,  $X = \sum_{b \in B} x_{b,1} \otimes \cdots \otimes x_{b,k}$  with a finite set  $B$  and vectors  $x_{b,i} \in V$ . Let

$$V_X := \text{span} \{x_{b,i} \mid b \in B, i \in \{1, \dots, k\}\} \text{ and } V_{X_0} := \{v \in V_X \mid \|v\|_\alpha = 0\}.$$

Construct a complementary linear subspace  $V_{\tilde{X}}$  of  $V_{X_0}$  in  $V_X$ , then we can also assume without loss of generality that  $x_{b,i} \in V_{X_0} \cup V_{\tilde{X}}$  for all  $b \in B$  and  $i \in \{1, \dots, k\}$ . Note that  $V_X, V_{X_0}$  and  $V_{\tilde{X}}$  are all finite-dimensional. Now define  $A := \{a \in B \mid \exists_{n \in \{1, \dots, k\}}: x_{a,n} \in V_{X_0}\}$  and  $X_0 := \sum_{a \in A} x_{a,1} \otimes \cdots \otimes x_{a,k}$ , then  $\prod_{n=1}^k \|x_{a,n}\|_\alpha = 0$  by construction and so  $\|X_0\|_\alpha^\bullet = 0$  and  $\|X - X_0\|_\alpha^\bullet = \|X\|_\alpha^\bullet$ . Restricted to  $V_{\tilde{X}}$ , the positive Hermitian form  $\langle \cdot | \cdot \rangle_\alpha$  is even positive definite, i.e. an inner product. Let  $d := \dim(V_{\tilde{X}})$  and  $e \in V^d$  be an  $\langle \cdot | \cdot \rangle_\alpha$ -orthonormal base of  $V_{\tilde{X}}$ . Define  $\tilde{X} := X - X_0$ , then  $\tilde{X} = \sum_{a' \in \{1, \dots, d\}^k} X^{a'} e_{a'_1} \otimes \cdots \otimes e_{a'_k}$  with complex coefficients  $X^{a'}$  and

$$\|X\|_\alpha^{\bullet,2} = \|\tilde{X}\|_\alpha^{\bullet,2} = \sum_{a' \in \{1, \dots, d\}^k} |X^{a'}|^2 \|e_{a'_1} \otimes \cdots \otimes e_{a'_k}\|_\alpha^{\bullet,2} = \sum_{a' \in \{1, \dots, d\}^k} |X^{a'}|^2 k!. \quad \square$$

On the locally convex space  $\mathcal{T}^\bullet(V)$ , the tensor product is indeed continuous and  $(\mathcal{T}^\bullet(V), \otimes)$  is a locally convex algebra. In order to see this, we are going to prove the continuity of the following function:

**Definition 2.4** We define the map  $\mu_\otimes: \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \rightarrow \mathcal{T}^\bullet(V)$  by

$$X \otimes_\pi Y \mapsto \mu_\otimes(X \otimes_\pi Y) := X \otimes Y. \quad (2.4)$$

Algebraically,  $\mu_\otimes$  is of course just the product of the tensor algebra. The emphasize lies here on the topologies involved:  $\otimes_\pi$  denotes the projective tensor product. We recall that the topology on  $\mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$  is described by the seminorms  $\|\cdot\|_{\alpha \otimes_\pi \beta}^\bullet: \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \rightarrow [0, \infty[$

$$Z \mapsto \|Z\|_{\alpha \otimes_\pi \beta}^\bullet := \inf \sum_{i \in I} \|X_i\|_\alpha^\bullet \|Y_i\|_\beta^\bullet, \quad (2.5)$$

where the infimum runs over all possibilities to express  $Z$  as a sum  $Z = \sum_{i \in I} X_i \otimes_\pi Y_i$  indexed by a finite set  $I$  and  $\|\cdot\|_\alpha^\bullet, \|\cdot\|_\beta^\bullet$  run over all extensions of continuous Hilbert seminorms on  $V$ . The only property of the projective tensor product relevant for our purposes is the following lemma, which is a direct result of the definition of the seminorms  $\|\cdot\|_{\alpha \otimes_\pi \beta}^\bullet$ :

**Lemma 2.5** *Let  $W$  be a locally convex space,  $p$  a continuous seminorm on  $W$  and  $\|\cdot\|_\alpha, \|\cdot\|_\beta \in \mathcal{P}_V$ . Let  $\Phi: \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \rightarrow W$  be a linear map. Then the two statements*

*i.)  $p(\Phi(X \otimes_\pi Y)) \leq \|X\|_\alpha^\bullet \|Y\|_\beta^\bullet$  for all  $X, Y \in \mathcal{T}^\bullet(V)$*

*ii.)  $p(\Phi(Z)) \leq \|Z\|_{\alpha \otimes_\pi \beta}^\bullet$  for all  $Z \in \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$*

*are equivalent. Continuity of the bilinear map  $\mathcal{T}^\bullet(V) \times \mathcal{T}^\bullet(V) \ni (X, Y) \mapsto \Phi(X \otimes_\pi Y) \in W$  is therefore equivalent to continuity of  $\Phi$ .*

**Proposition 2.6** *The linear map  $\mu_\otimes$  is continuous and the estimate*

$$\|\mu_\otimes(Z)\|_\gamma^\bullet \leq \|Z\|_{2\gamma \otimes_\pi 2\gamma}^\bullet \quad (2.6)$$

*holds for all  $Z \in \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$  and all  $\|\cdot\|_\gamma \in \mathcal{P}_V$ . Moreover, all  $X \in \mathcal{T}^k(V)$  and  $Y \in \mathcal{T}^\ell(V)$  with  $k, \ell \in \mathbb{N}_0$  fulfill for all  $\|\cdot\|_\gamma \in \mathcal{P}_V$  the estimate*

$$\|\mu_\otimes(X \otimes_\pi Y)\|_\gamma^\bullet \leq \binom{k+\ell}{k}^{1/2} \|X\|_\gamma^\bullet \|Y\|_\gamma^\bullet. \quad (2.7)$$

PROOF: Let  $X \in \mathcal{T}^k(V)$  and  $Y \in \mathcal{T}^\ell(V)$  with  $k, \ell \in \mathbb{N}_0$  be given. Then

$$\|X \otimes Y\|_\gamma^\bullet = \sqrt{\langle X \otimes Y | X \otimes Y \rangle_\gamma^\bullet} = \binom{k+\ell}{k}^{1/2} \|X\|_\gamma^\bullet \|Y\|_\gamma^\bullet$$

holds. It now follows for all  $X, Y \in \mathcal{T}^\bullet(V)$  that

$$\begin{aligned} \|X \otimes Y\|_\gamma^{\bullet,2} &= \sum_{m=0}^{\infty} \|\langle X \otimes Y \rangle_m\|_\gamma^{\bullet,2} \\ &\leq \sum_{m=0}^{\infty} \left( \sum_{n=0}^m \|\langle X \rangle_{m-n} \otimes \langle Y \rangle_n\|_\gamma^\bullet \right)^2 \\ &= \sum_{m=0}^{\infty} \left( \sum_{n=0}^m \binom{m}{n}^{\frac{1}{2}} \|\langle X \rangle_{m-n}\|_\gamma^\bullet \|\langle Y \rangle_n\|_\gamma^\bullet \right)^2 \\ &= \sum_{m=0}^{\infty} \left( \sum_{n=0}^m \left( \binom{m}{n} \frac{1}{2^m} \right)^{\frac{1}{2}} \|\langle X \rangle_{m-n}\|_{2\gamma}^{\bullet,2} \|\langle Y \rangle_n\|_{2\gamma}^{\bullet,2} \right)^2 \\ &\stackrel{\text{CS}}{\leq} \sum_{m=0}^{\infty} \left( \sum_{n=0}^m \binom{m}{n} \frac{1}{2^m} \right) \left( \sum_{n=0}^m \|\langle X \rangle_{m-n}\|_{2\gamma}^{\bullet,2} \|\langle Y \rangle_n\|_{2\gamma}^{\bullet,2} \right) \\ &= \|X\|_{2\gamma}^{\bullet,2} \|Y\|_{2\gamma}^{\bullet,2}, \end{aligned}$$

by the Cauchy-Schwarz (CS) inequality. □

## 2.2 Symmetrisation

The star product will be defined on the symmetric tensor algebra with undeformed product  $X \vee Y := \mathcal{S}^\bullet(X \otimes Y)$  for  $X, Y \in \mathcal{S}^\bullet(V)$ , which is indeed continuous:

**Proposition 2.7** *The symmetrisation operator is continuous and fulfills  $\|\mathcal{S}^\bullet X\|_\gamma^\bullet \leq \|X\|_\gamma^\bullet$  for all  $X \in \mathcal{T}^\bullet(V)$  and  $\|\cdot\|_\gamma \in \mathcal{P}_V$ .*

PROOF: From Definition 2.1 it is clear that  $\langle X^\sigma | Y^\sigma \rangle_\gamma^\bullet = \langle X | Y \rangle_\gamma^\bullet$  for all  $k \in \mathbb{N}_0$ ,  $X, Y \in \mathcal{T}^k(V)$  and  $\sigma \in \mathfrak{S}_k$ , because this holds for all simple tensors and because both sides are (anti-)linear in  $X$  and  $Y$ . Therefore  $\|X^\sigma\|_\gamma^\bullet = \|X\|_\gamma^\bullet$  and  $\|\mathcal{S}^k X\|_\gamma^\bullet \leq \|X\|_\gamma^\bullet$  and we get the desired estimate

$$\|\mathcal{S}^\bullet X\|_\gamma^{\bullet,2} = \sum_{k=0}^{\infty} \|\mathcal{S}^k \langle X \rangle_k\|_\gamma^{\bullet,2} \leq \sum_{k=0}^{\infty} \|\langle X \rangle_k\|_\gamma^{\bullet,2} = \|X\|_\gamma^{\bullet,2}$$

on  $\mathcal{T}^\bullet(V)$ . □

Analogously to  $\mu_\otimes$  we define the linear map  $\mu_\vee := \mathcal{S}^\bullet \circ \mu_\otimes : \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \rightarrow \mathcal{T}^\bullet(V)$ . Then the restriction of  $\mu_\vee$  to  $\mathcal{S}^\bullet(V)$  describes the symmetric tensor product  $\vee$  and Propositions 2.6 and 2.7 yield:

**Corollary 2.8** *The linear map  $\mu_\vee$  is continuous and the estimate  $\|\mu_\vee(Z)\|_\gamma^\bullet \leq \|Z\|_{2\gamma \otimes_\pi 2\gamma}^\bullet$  holds for all  $Z \in \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$  and all  $\|\cdot\|_\gamma \in \mathcal{P}_V$ .*

### 2.3 The Star Product

The following star product is based on a bilinear form and generalizes the usual exponential-type star product like the Weyl-Moyal or Wick star product, see e.g. [22, Chap. 5], to arbitrary dimensions:

**Definition 2.9** *For every continuous bilinear form  $\Lambda$  on  $V$  we define the product  $\mu_{\star_\Lambda} : \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \rightarrow \mathcal{T}^\bullet(V)$  by*

$$X \otimes_\pi Y \mapsto \mu_{\star_\Lambda}(X \otimes_\pi Y) := \sum_{t=0}^{\infty} \frac{1}{t!} \mu_\vee \left( (P_\Lambda)^t (X \otimes_\pi Y) \right), \quad (2.8)$$

where the linear map  $P_\Lambda : \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \rightarrow \mathcal{T}^{\bullet-1}(V) \otimes_\pi \mathcal{T}^{\bullet-1}(V)$  is given on factorizing tensors of degree  $k, \ell \in \mathbb{N}$  by

$$P_\Lambda((x_1 \otimes \cdots \otimes x_k) \otimes_\pi (y_1 \otimes \cdots \otimes y_\ell)) := k\ell \Lambda(x_k, y_1)(x_1 \otimes \cdots \otimes x_{k-1}) \otimes_\pi (y_2 \otimes \cdots \otimes y_\ell) \quad (2.9)$$

for all  $x \in V^k$  and  $y \in V^\ell$ . Moreover, we define the product  $\star_\Lambda$  on  $\mathcal{S}^\bullet(V)$  as the bilinear map described by the restriction of  $\mu_{\star_\Lambda}$  to  $\mathcal{S}^\bullet(V)$ .

Note that these definitions of  $P_\Lambda$  and  $\star_\Lambda$  coincide (algebraically) on  $\mathcal{S}^\bullet(V)$  with the ones in [23, Eq. (2.13) and (2.19)], evaluated at a fixed value for  $\nu$  in the truly (not graded) symmetric case  $V = V_0$ . Note that with our convention the deformation parameter  $\hbar$  is already part of  $\Lambda$ .

We are now going to prove the continuity of  $\star_\Lambda$ . Therefore we note that continuity of  $\Lambda$  means that there exist  $\|\cdot\|_\alpha, \|\cdot\|_\beta \in \mathcal{P}_V$  such that  $|\Lambda(v, w)| \leq \|v\|_\alpha \|w\|_\beta$  holds for all  $v, w \in V$ . So the set

$$\mathcal{P}_{V,\Lambda} := \left\{ \|\cdot\|_\gamma \in \mathcal{P}_V \mid |\Lambda(v, w)| \leq \|v\|_\gamma \|w\|_\gamma \text{ for all } v, w \in V \right\} \quad (2.10)$$

contains at least all continuous Hilbert seminorms on  $V$  that dominate  $\|\cdot\|_{\alpha+\beta}$ . Thus this set is cofinal in  $\mathcal{P}_V$ .

**Lemma 2.10** *Let  $\Lambda$  be a continuous bilinear form on  $V$ , let  $\|\cdot\|_\alpha, \|\cdot\|_\beta \in \mathcal{P}_{V,\Lambda}$  as well as  $k, \ell \in \mathbb{N}_0$  and  $X \in \mathcal{T}^k(V)$ ,  $Y \in \mathcal{T}^\ell(V)$  be given. Then*

$$\|P_\Lambda(X \otimes_\pi Y)\|_{\alpha \otimes_\pi \beta}^\bullet \leq \sqrt{k\ell} \|X\|_\alpha^\bullet \|Y\|_\beta^\bullet. \quad (2.11)$$



PROOF: If  $k = 0$  or  $\ell = 0$  this is clearly true, so assume  $k, \ell \in \mathbb{N}$ . We use Lemma 2.3 to construct  $X_0 = \sum_{a \in A} x_{a,1} \otimes \cdots \otimes x_{a,k}$  and  $\tilde{X} = \sum_{a' \in \{1, \dots, c\}^k} X^{a'} e_{a'_1} \otimes \cdots \otimes e_{a'_k}$  with respect to  $\langle \cdot | \cdot \rangle_\alpha$  as well as  $Y_0 = \sum_{b \in B} y_{b,1} \otimes \cdots \otimes y_{b,\ell}$  and  $\tilde{Y} = \sum_{b' \in \{1, \dots, d\}^\ell} Y^{b'} f_{b'_1} \otimes \cdots \otimes f_{b'_\ell}$  with respect to  $\langle \cdot | \cdot \rangle_\beta$ . Then

$$\|P_\Lambda((X_0 + \tilde{X}) \otimes_\pi (Y_0 + \tilde{Y}))\|_{\alpha \otimes_\pi \beta}^\bullet \leq \|P_\Lambda(\tilde{X} \otimes_\pi \tilde{Y})\|_{\alpha \otimes_\pi \beta}^\bullet,$$

because

$$\begin{aligned} \|P_\Lambda((\xi_1 \otimes \cdots \otimes \xi_k) \otimes_\pi (\eta_1 \otimes \cdots \otimes \eta_\ell))\|_{\alpha \otimes_\pi \beta}^\bullet &= k\ell |\Lambda(\xi_k, \eta_1)| \|\xi_1 \otimes \cdots \otimes \xi_{k-1}\|_\alpha^\bullet \|\eta_2 \otimes \cdots \otimes \eta_\ell\|_\beta^\bullet \\ &= 0 \end{aligned}$$

for all  $\xi \in V^k$ ,  $\eta \in V^\ell$  for which there is at least one  $m \in \{1, \dots, k\}$  with  $\|\xi_m\|_\alpha = 0$  or one  $n \in \{1, \dots, \ell\}$  with  $\|\eta_n\|_\beta = 0$ . On the subspaces  $V_{\tilde{X}} = \text{span}\{e_1, \dots, e_c\}$  and  $V_{\tilde{Y}} = \text{span}\{f_1, \dots, f_d\}$  of  $V$ , the bilinear form  $\Lambda$  is described by a matrix  $\Omega \in \mathbb{C}^{c \times d}$  with entries  $\Omega_{gh} = \Lambda(e_g, f_h)$ . By using a singular value decomposition we can even assume without loss of generality that all off-diagonal entries of  $\Omega$  vanish. We also note that  $|\Omega_{gg}| = |\Lambda(e_g, f_g)| \leq \|e_g\|_\alpha \|f_g\|_\beta \leq 1$ . This gives the desired estimate

$$\begin{aligned} &\|P_\Lambda(X \otimes_\pi Y)\|_{\alpha \otimes_\pi \beta}^\bullet \\ &\leq \|P_\Lambda(\tilde{X} \otimes_\pi \tilde{Y})\|_{\alpha \otimes_\pi \beta}^\bullet \\ &= \left\| \sum_{a' \in \{1, \dots, c\}^k} \sum_{b' \in \{1, \dots, d\}^\ell} X^{a'} Y^{b'} P_\Lambda((e_{a'_1} \otimes \cdots \otimes e_{a'_k}) \otimes_\pi (f_{b'_1} \otimes \cdots \otimes f_{b'_\ell})) \right\|_{\alpha \otimes_\pi \beta}^\bullet \\ &= k\ell \left\| \sum_{r=1}^{\min\{c,d\}} \sum_{\substack{\tilde{a}' \in \{1, \dots, c\}^{k-1} \\ \tilde{b}' \in \{1, \dots, d\}^{\ell-1}}} X^{(\tilde{a}', r)} Y^{(r, \tilde{b}')} \Omega_{rr} (e_{\tilde{a}'_1} \otimes \cdots \otimes e_{\tilde{a}'_{k-1}}) \otimes_\pi (f_{\tilde{b}'_1} \otimes \cdots \otimes f_{\tilde{b}'_{\ell-1}}) \right\|_{\alpha \otimes_\pi \beta}^\bullet \\ &\leq k\ell \sum_{r=1}^{\min\{c,d\}} \left\| \sum_{\tilde{a}' \in \{1, \dots, c\}^{k-1}} X^{(\tilde{a}', r)} e_{\tilde{a}'_1} \otimes \cdots \otimes e_{\tilde{a}'_{k-1}} \right\|_\alpha^\bullet \left\| \sum_{\tilde{b}' \in \{1, \dots, d\}^{\ell-1}} Y^{(r, \tilde{b}')} f_{\tilde{b}'_1} \otimes \cdots \otimes f_{\tilde{b}'_{\ell-1}} \right\|_\beta^\bullet \\ &\stackrel{\text{CS}}{\leq} \sqrt{k\ell} \|X\|_\alpha^\bullet \|Y\|_\beta^\bullet, \end{aligned}$$

where we have used in the last line after applying the Cauchy-Schwarz inequality that

$$\begin{aligned} \sum_{r=1}^{\min\{c,d\}} \left\| \sum_{\tilde{a}' \in \{1, \dots, c\}^{k-1}} X^{(\tilde{a}', r)} e_{\tilde{a}'_1} \otimes \cdots \otimes e_{\tilde{a}'_{k-1}} \right\|_\alpha^{\bullet, 2} &= \sum_{r=1}^{\min\{c,d\}} \sum_{\tilde{a}' \in \{1, \dots, c\}^{k-1}} |X^{(\tilde{a}', r)}|^2 (k-1)! \\ &\leq \frac{1}{k} \|X\|_\alpha^{\bullet, 2} \end{aligned}$$

and analogously for  $Y$ . □

**Proposition 2.11** *Let  $\Lambda$  be a continuous bilinear form on  $V$ , then the function  $P_\Lambda$  is continuous and fulfills the estimate*

$$\|(P_\Lambda)^t(Z)\|_{\alpha \otimes_\pi \beta}^\bullet \leq \frac{c}{c-1} \frac{t!}{c^t} \|Z\|_{2c\alpha \otimes_\pi 2c\beta}^\bullet \quad (2.12)$$

for all  $c > 1$ , all  $t \in \mathbb{N}_0$ , all seminorms  $\|\cdot\|_\alpha, \|\cdot\|_\beta \in \mathcal{P}_{V, \Lambda}$ , and all  $Z \in \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$ .

PROOF: Let  $X, Y \in \mathcal{T}^\bullet(V)$  be given, then the previous Lemma 2.10 together with Lemma 2.5 yields

$$\|(P_\Lambda)^t(X \otimes_\pi Y)\|_{\alpha \otimes_\pi \beta}^\bullet \leq \sum_{k, \ell=0}^{\infty} \|(P_\Lambda)^t(\langle X \rangle_{k+t} \otimes_\pi \langle Y \rangle_{\ell+t})\|_{\alpha \otimes_\pi \beta}^\bullet$$

$$\begin{aligned}
&\leq t! \sum_{k,\ell=0}^{\infty} \binom{k+t}{t}^{\frac{1}{2}} \binom{\ell+t}{t}^{\frac{1}{2}} \|\langle X \rangle_{k+t}\|_{\alpha}^{\bullet} \|\langle Y \rangle_{\ell+t}\|_{\beta}^{\bullet} \\
&\leq t! \sum_{k,\ell=0}^{\infty} \|\langle X \rangle_{k+t}\|_{2\alpha}^{\bullet} \|\langle Y \rangle_{\ell+t}\|_{2\beta}^{\bullet} \\
&= \frac{t!}{c^t} \sum_{k,\ell=0}^{\infty} \frac{1}{\sqrt{c}^{k+\ell}} \|\langle X \rangle_{k+t}\|_{2c\alpha}^{\bullet} \|\langle Y \rangle_{\ell+t}\|_{2c\beta}^{\bullet} \\
&\stackrel{\text{CS}}{\leq} \frac{t!}{c^t} \left( \sum_{k,\ell=0}^{\infty} \frac{1}{c^{k+\ell}} \right)^{\frac{1}{2}} \left( \sum_{k,\ell=0}^{\infty} \|\langle X \rangle_{k+t}\|_{2c\alpha}^{\bullet,2} \|\langle Y \rangle_{\ell+t}\|_{2c\beta}^{\bullet,2} \right)^{\frac{1}{2}} \\
&\leq \frac{c}{c-1} \frac{t!}{c^t} \|X\|_{2c\alpha}^{\bullet} \|Y\|_{2c\beta}^{\bullet}. \quad \square
\end{aligned}$$

**Lemma 2.12** *Let  $\Lambda$  be a continuous bilinear form on  $V$ , then  $\mu_{\star_{\Lambda}}$  is continuous and, given  $R > 1/2$ , the estimate*

$$\|\mu_{\star_{\Lambda}}(Z)\|_{\gamma}^{\bullet} \leq \sum_{t=0}^{\infty} \frac{1}{t!} \left\| \mu_{\vee} \left( (P_{z\Lambda})^t(Z) \right) \right\|_{\gamma}^{\bullet} \leq \frac{4R}{2R-1} \|Z\|_{8R\gamma \otimes_{\pi} 8R\gamma}^{\bullet} \quad (2.13)$$

holds for all  $\|\cdot\|_{\gamma} \in \mathcal{P}_{V,\Lambda}$ , all  $Z \in \mathcal{T}^{\bullet}(V) \otimes_{\pi} \mathcal{T}^{\bullet}(V)$  and all  $z \in \mathbb{C}$  with  $|z| \leq R$ .

PROOF: The first estimate is just the triangle-inequality. By combining Corollary 2.8 and Proposition 2.11 with  $c = 2R$  we get the second estimate

$$\begin{aligned}
\sum_{t=0}^{\infty} \frac{1}{t!} \left\| \mu_{\vee} \left( (P_{z\Lambda})^t(Z) \right) \right\|_{\gamma}^{\bullet} &\leq \sum_{t=0}^{\infty} \frac{|z|^t}{t!} \left\| (P_{\Lambda})^t(Z) \right\|_{2\gamma \otimes_{\pi} 2\gamma}^{\bullet} \\
&\leq \frac{2R}{2R-1} \sum_{t=0}^{\infty} \frac{1}{2^t} \|Z\|_{8R\gamma \otimes_{\pi} 8R\gamma} \\
&= \frac{4R}{2R-1} \|Z\|_{8R\gamma \otimes_{\pi} 8R\gamma}. \quad \square
\end{aligned}$$

This estimate immediately leads to:

**Theorem 2.13** *Let  $\Lambda$  be a continuous bilinear form on  $V$ , then the product  $\star_{\Lambda}$  is continuous and  $(\mathcal{S}^{\bullet}(V), \star_{\Lambda})$  is a locally convex algebra. Moreover, for fixed tensors  $X, Y$  from the completion  $\mathcal{S}^{\bullet}(V)^{\text{cpl}}$ , the product  $X \star_{z\Lambda} Y$  converges absolutely and locally uniformly in  $z \in \mathbb{C}$  and thus depends holomorphically on  $z$ .*

Note that the above estimate also shows that  $(\mathcal{S}^{\bullet}(V), \star_{z\Lambda})$  describes a holomorphic deformation (as defined in [18]) of the locally convex algebra  $(\mathcal{S}^{\bullet}(V), \vee)$ . However, in the following we will examine the star product for fixed values of both  $\Lambda$  and  $z$  and therefore can absorb the deformation parameter  $z$  in the bilinear form  $\Lambda$ .

### 3 Properties of the Star Product

In this section we want to examine some properties of the products  $\star_{\Lambda}$ , namely how the topology on  $\mathcal{S}^{\bullet}(V)$  can be characterized by demanding that certain algebraic operations are continuous, which products are equivalent, how to transform  $\mathcal{S}^{\bullet}(V)$  to a space of complex functions, the existence of continuous positive linear functionals and whether or not some exponentials of elements in  $\mathcal{S}^{\bullet}(V)$  exist and which elements are represented by essentially self-adjoint operators via GNS-construction. At some points we will also work with the completion  $\mathcal{S}^{\bullet}(V)^{\text{cpl}}$  of  $\mathcal{S}^{\bullet}(V)$  and therefore note that the previous constructions and results extend to  $\mathcal{S}^{\bullet}(V)^{\text{cpl}}$  by continuity.

### 3.1 Characterization of the Topology

We are going to show that the topology on  $\mathcal{S}^\bullet(V)$  that was defined in the last section in a rather unmotivated way is – under some additional assumptions – the coarsest possible one. More precisely, we want to express the extensions of positive Hermitian forms with the help of suitable star products. Due to the sesquilinearity of positive Hermitian forms, this is only possible if we also have an antilinear structure on  $\mathcal{S}^\bullet(V)$ , so we construct a  $*$ -involution.

There is clearly one and only one possibility to extend an antilinear involution  $\bar{\cdot}$  on  $V$  to a  $*$ -involution  $*$ :  $\mathcal{T}^\bullet(V) \rightarrow \mathcal{T}^\bullet(V)$  on the tensor algebra over  $V$ , namely by  $(x_1 \otimes \cdots \otimes x_k)^* := \bar{x}_k \otimes \cdots \otimes \bar{x}_1$  for all  $k \in \mathbb{N}$  and  $x \in V^k$  and antilinear extension. Its restriction to  $\mathcal{S}^\bullet(V)$  gives a  $*$ -involution on  $(\mathcal{S}^\bullet(V), \vee)$ .

**Proposition 3.1** *Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$ , then the induced  $*$ -involution on  $\mathcal{T}^\bullet(V)$  is also continuous.*

PROOF: For  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_V$  define the continuous positive Hermitian form  $V^2 \ni (v, w) \mapsto \langle v | w \rangle_{\alpha^*} := \overline{\langle \bar{v} | \bar{w} \rangle_\alpha}$ . Then  $\langle X^* | Y^* \rangle_{\alpha^*} = \overline{\langle X | Y \rangle_\alpha}$  and in particular  $\|X^*\|_{\alpha^*} = \|X\|_\alpha$  for all  $X, Y \in \mathcal{T}^\bullet(V)$  because this is clearly true for simple tensors and because both sides are (anti-)linear in  $X$  and  $Y$ .  $\square$

For certain bilinear forms  $\Lambda$  on  $V$  we can also show that  $*$  is a  $*$ -involution of  $\star_\Lambda$ , which is of course not a new result:

**Definition 3.2** *Let  $\bar{\cdot}$ :  $V \rightarrow V$  be a continuous antilinear involution on  $V$ . For every continuous bilinear form  $\Lambda: V \times V \rightarrow \mathbb{C}$  we define its conjugate  $\Lambda^*$  by  $\Lambda^*(v, w) := \overline{\Lambda(\bar{w}, \bar{v})}$ , which is again a continuous bilinear form on  $V$ . We say that  $\Lambda$  is Hermitian if  $\Lambda = \Lambda^*$  holds.*

Note that the bilinear form  $(v, w) \mapsto \Lambda(v, w)$  is Hermitian if and only if the sesquilinear form  $(v, w) \mapsto \Lambda(\bar{v}, w)$  is Hermitian. The typical example of a complex vector space  $V$  with antilinear involution  $\bar{\cdot}$  is that  $V = W \otimes \mathbb{C}$  is the complexification of a real vector space  $W$  with the canonical involution  $\bar{w \otimes \lambda} := w \otimes \bar{\lambda}$ . In this case, every bilinear form  $\Lambda$  on  $V$  is fixed by two bilinear forms  $\Lambda_r, \Lambda_i: W \times W \rightarrow \mathbb{R}$ , the restriction of the real- and imaginary part of  $\Lambda$  to the real subspace  $W \cong W \otimes 1$  of  $V$ , and  $\Lambda$  is Hermitian if and only if  $\Lambda_r$  is symmetric and  $\Lambda_i$  antisymmetric. Similarly to [23, Prop. 3.25] we get:

**Proposition 3.3** *Let  $\bar{\cdot}$ :  $V \rightarrow V$  be a continuous antilinear involution and  $\Lambda$  a continuous bilinear form on  $V$ . Then  $(X \star_\Lambda Y)^* = Y^* \star_{\Lambda^*} X^*$  holds for all  $X, Y \in \mathcal{S}^\bullet(V)$ . Consequently, if  $\Lambda$  is Hermitian, then  $(\mathcal{S}^\bullet(V), \star_\Lambda, *)$  is a locally convex  $*$ -algebra.*

PROOF: The identities  $* \circ \mathcal{S}^\bullet = \mathcal{S}^\bullet \circ *$  and  $* \circ \mu_\otimes = \mu_\otimes \circ \tau \circ (* \otimes_\pi *)$ , with  $\tau: \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V) \rightarrow \mathcal{T}^\bullet(V) \otimes_\pi \mathcal{T}^\bullet(V)$  defined as  $\tau(X \otimes_\pi Y) := Y \otimes_\pi X$ , can easily be checked on simple tensors, so  $* \circ \mu_\vee = \mu_\vee \circ \tau \circ (* \otimes_\pi *)$ . Combining this with  $\tau \circ (* \otimes_\pi *) \circ P_\Lambda = P_{\Lambda^*} \circ \tau \circ (* \otimes_\pi *)$  on symmetric tensors, which again can easily be checked on simple symmetric tensors, yields the desired result.  $\square$

**Lemma 3.4** *Let  $\bar{\cdot}$ :  $V \rightarrow V$  be a continuous antilinear involution. For every  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_V$  we define a continuous bilinear form  $\Lambda_\alpha$  on  $V$  by  $\Lambda_\alpha(v, w) := \langle \bar{v} | w \rangle_\alpha$  for all  $v, w \in V$ , then  $\Lambda_\alpha$  is Hermitian and the identities*

$$\sum_{t=0}^{\infty} \frac{1}{t!} \mu_\otimes \left( (P_{\Lambda_\alpha})^t (\langle X^* \rangle_t \otimes_\pi \langle Y \rangle_t) \right) = \langle X | Y \rangle_{\alpha^*} \quad (3.1)$$

and

$$\langle \mu_{\star_{\Lambda_\alpha}} (X^* \otimes_\pi Y) \rangle_0 = \langle X | Y \rangle_{\alpha^*} \quad (3.2)$$

hold for all  $X, Y \in \mathcal{T}^\bullet(V)$ .

PROOF: Clearly,  $\Lambda_\alpha$  is Hermitian because  $\langle \cdot | \cdot \rangle_\alpha$  is Hermitian. Then (3.2) follows directly from (3.1) because of the grading of  $\mu_\vee$  and  $P_{\Lambda_\alpha}$ . For proving (3.1) it is sufficient to check it for factorizing tensors of the same degree, because both sides are (anti-)linear in  $X$  and  $Y$  and vanish if  $X$  and  $Y$  are homogeneous of different degree. If  $X$  and  $Y$  are of degree 0 then (3.1) is clearly fulfilled. Otherwise we get

$$\begin{aligned}
& \frac{1}{k!} \mu_\otimes \left( (P_{\Lambda_\alpha})^k ((x_1 \otimes \cdots \otimes x_k)^* \otimes_\pi (y_1 \otimes \cdots \otimes y_k)) \right) \\
&= \frac{1}{k!} \mu_\otimes \left( (P_{\Lambda_\alpha})^k ((\bar{x}_k \otimes \cdots \otimes \bar{x}_1) \otimes_\pi (y_1 \otimes \cdots \otimes y_k)) \right) \\
&= \frac{1}{k!} \mu_\otimes \left( (1 \otimes_\pi 1) (k!)^2 \prod_{m=1}^k \Lambda_\alpha(\bar{x}_m, y_m) \right) \\
&= k! \prod_{m=1}^k \Lambda_\alpha(\bar{x}_m, y_m) \\
&= k! \prod_{m=1}^k \langle x_m | y_m \rangle_\alpha \\
&= \langle x_1 \otimes \cdots \otimes x_k | y_1 \otimes \cdots \otimes y_k \rangle_\alpha^\bullet.
\end{aligned}$$

□

**Theorem 3.5** *The topology on  $\mathcal{S}^\bullet(V)$  is the coarsest locally convex one that makes all star products  $\star_\Lambda$  for all continuous and Hermitian bilinear forms  $\Lambda$  on  $V$  as well as the  $*$ -involution and the projection  $\langle \cdot \rangle_0$  onto the scalars continuous. In addition we have for all  $X, Y \in \mathcal{S}^\bullet(V)$  and all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_V$*

$$\langle X^* \star_{\Lambda_\alpha} Y \rangle_0 = \langle X | Y \rangle_\alpha^\bullet, \quad (3.3)$$

with  $\Lambda_\alpha$  as in Lemma 3.4.

PROOF: We have already shown the continuity of the star product and of the  $*$ -involution, the continuity of  $\langle \cdot \rangle_0$  is clear. Conversely, if these three functions are continuous, their compositions yield the extensions of all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_V$  which then have to be continuous. Then (3.2) gives (3.3) for symmetric tensors  $X$  and  $Y$ . □

### 3.2 Equivalence of Star Products

Next we want to examine the usual equivalence transformations between star products, given by exponentials of a Laplace operator (see [23] for the algebraic background).

**Definition 3.6** *Let  $b: V \times V \rightarrow \mathbb{C}$  be a symmetric bilinear form on  $V$ , i.e.  $b(v, w) = b(w, v)$  for all  $v, w \in V$ . Then we define the Laplace operator  $\Delta_b: \mathcal{T}^\bullet(V) \rightarrow \mathcal{T}^{\bullet-2}(V)$  as the linear map given on simple tensors of degree  $k \in \mathbb{N} \setminus \{1\}$  by*

$$\Delta_b(x_1 \otimes \cdots \otimes x_k) := \frac{k(k-1)}{2} b(x_1, x_2) x_3 \otimes \cdots \otimes x_k. \quad (3.4)$$

Note that  $\Delta_b$  can be restricted to symmetric tensors on which it coincides with the Laplace operator from [23, Eq. (2.31)]. However, there is no need for  $\Delta_b$  to be continuous even if  $b$  is continuous, because the Hilbert tensor product in general does not allow the extension of all continuous multilinear forms. Note that this is very different from the approach taken in [23] where the projective tensor product was used: this guaranteed the continuity of the Laplace operator directly for all continuous bilinear forms.

For the restriction of  $\Delta_b$  to  $\mathcal{S}^2(V)$ , continuity is equivalent to the existence of a  $\|\cdot\|_\alpha \in \mathcal{P}_V$  that fulfills  $|\Delta_b X| \leq \|X\|_\alpha^\bullet$  for all  $X \in \mathcal{S}^2(V)$ . This motivates the following:

**Definition 3.7** A bilinear form of Hilbert-Schmidt type on  $V$  is a bilinear form  $b: V \times V \rightarrow \mathbb{C}$  for which there is a seminorm  $\|\cdot\|_\alpha \in \mathcal{P}_V$  such that the following two conditions are fulfilled:

- i.) If  $\|v\|_\alpha = 0$  or  $\|w\|_\alpha = 0$  for vectors  $v, w \in V$ , then  $b(v, w) = 0$ .
- ii.) For every tuple of  $\langle \cdot | \cdot \rangle_\alpha$ -orthonormal vectors  $e \in V^d$ ,  $d \in \mathbb{N}$ , the estimate

$$\sum_{i,j=1}^d |b(e_i, e_j)|^2 \leq 1 \quad (3.5)$$

holds.

For such a bilinear form of Hilbert-Schmidt type  $b$  we define  $\mathcal{P}_{V,b,HS}$  as the set of all  $\|\cdot\|_\alpha \in \mathcal{P}_V$  that fulfill these two conditions.

We can characterize the bilinear forms of Hilbert-Schmidt type in the following way:

**Proposition 3.8** Let  $b$  be a symmetric bilinear form on  $V$  and  $\|\cdot\|_\alpha \in \mathcal{P}_V$ , then the following two statements are equivalent:

- i.) The bilinear form  $b$  is of Hilbert-Schmidt type and  $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$ .
- ii.) The estimate  $|\Delta_b X| \leq 2^{-1/2} \|X\|_\alpha^\bullet$  holds for all  $X \in \mathcal{S}^2(V)$ .

Moreover, if this holds then  $\|\cdot\|_\alpha \in \mathcal{P}_{V,b}$  and  $b$  is continuous.

PROOF: If the first point holds, let  $X \in \mathcal{T}^2(V)$  be given. Construct  $X_0 = \sum_{a \in A} x_{a,1} \otimes x_{a,2}$  and  $\tilde{X} = \sum_{a'_1, a'_2=1}^d X^{a'_1, a'_2} e_{a'_1} \otimes e_{a'_2} \in \mathcal{T}^2(V)$  like in Lemma 2.3. Then  $b(x_{a,1}, x_{a,2}) = 0$  for all  $a \in A$  because  $\|x_{a,1}\|_\alpha = 0$  or  $\|x_{a,2}\|_\alpha = 0$ . Moreover,

$$\begin{aligned} |\Delta_b X| &\leq \left| \sum_{a'_1, a'_2=1}^d X^{a'_1, a'_2} b(e_{a'_1}, e_{a'_2}) \right| \\ &\stackrel{\text{CS}}{\leq} \left( \sum_{a'_1, a'_2=1}^d |X^{a'_1, a'_2}|^2 \right)^{\frac{1}{2}} \left( \sum_{a'_1, a'_2=1}^d |b(e_{a'_1}, e_{a'_2})|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{2}} \|X\|_\alpha^\bullet \end{aligned}$$

shows that the second point holds. Conversely, from the second point we get  $|b(v, w)| = |\Delta_b(v \vee w)| \leq 2^{-1/2} \|v \vee w\|_\alpha^\bullet \leq \|v\|_\alpha \|w\|_\alpha$  for all  $v, w \in V$ . Hence  $\|\cdot\|_\alpha \in \mathcal{P}_{V,b}$ , the bilinear form  $b$  is continuous, and  $b(v, w) = 0$  if one of  $v$  or  $w$  is in the kernel of  $\|\cdot\|_\alpha$ . Moreover, given an  $\langle \cdot | \cdot \rangle_\alpha$ -orthonormal set of vectors  $e \in V^d$ ,  $d \in \mathbb{N}$ , we define  $X := \sum_{i,j=1}^d \overline{b(e_i, e_j)} e_i \otimes e_j \in \mathcal{S}^2(V)$  and get

$$0 \leq \sum_{i,j=1}^d |b(e_i, e_j)|^2 = |\Delta_b X| \leq \frac{1}{\sqrt{2}} \|X\|_\alpha^\bullet = \left( \sum_{i,j=1}^d |b(e_i, e_j)|^2 \right)^{\frac{1}{2}},$$

which implies  $\sum_{i,j=1}^d |b(e_i, e_j)|^2 \leq 1$ . □

Note that this also implies that for a bilinear form of Hilbert-Schmidt type  $b$ , the set  $\mathcal{P}_{V,b,HS}$  is cofinal in  $\mathcal{P}_V$ , because if  $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$  and  $\|\cdot\|_\beta \geq \|\cdot\|_\alpha$ , then  $|\Delta_b X| \leq 2^{-1/2} \|X\|_\alpha^\bullet \leq 2^{-1/2} \|X\|_\beta^\bullet$  and so  $\|\cdot\|_\beta \in \mathcal{P}_{V,b,HS}$ .

As a consequence of the above characterization we see that a symmetric bilinear form  $b$  on  $V$  has to be of Hilbert-Schmidt type if we want  $\Delta_b$  to be continuous. We are going to show now that this is also sufficient:

**Proposition 3.9** *Let  $b$  be a symmetric bilinear form of Hilbert-Schmidt type on  $V$ , then the Laplace operator  $\Delta_b$  is continuous and fulfills the estimate*

$$\|(\Delta_b)^t X\|_\alpha^\bullet \leq \frac{\sqrt{(2t)!}}{(2r)^t} \|X\|_{2r\alpha}^\bullet \quad (3.6)$$

for all  $X \in \mathcal{T}^\bullet(V)$ ,  $t \in \mathbb{N}_0$ ,  $r \geq 1$ , and all  $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$ .

PROOF: First, let  $X \in \mathcal{T}^k(V)$ ,  $k \geq 2$ , and  $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$  be given. Construct  $X_0 = \sum_{a \in A} x_{a,1} \otimes \cdots \otimes x_{a,k}$  and  $\tilde{X} = \sum_{a' \in \{1, \dots, d\}^k} X^{a'} e_{a'_1} \otimes \cdots \otimes e_{a'_k}$  like in Lemma 2.3. Then again

$$\|\Delta_b X_0\|_\alpha^\bullet \leq \frac{k(k-1)\sqrt{(k-2)!}}{2} \sum_{a \in A} |b(x_{a_1}, x_{a_2})| \prod_{m=3}^k \|x_{a_m}\|_\alpha = 0$$

shows that  $\|\Delta_b X\|_\alpha^\bullet \leq \|\Delta_b \tilde{X}\|_\alpha^\bullet$ . For  $\tilde{X}$  we get:

$$\begin{aligned} \|\Delta_b \tilde{X}\|_\alpha^{\bullet,2} &= \left\| \frac{k(k-1)}{2} \sum_{a' \in \{1, \dots, d\}^k} X^{a'} b(e_{a'_1}, e_{a'_2}) e_{a'_3} \otimes \cdots \otimes e_{a'_k} \right\|_\alpha^{\bullet,2} \\ &= \frac{k^2(k-1)^2}{4} \sum_{\tilde{a}' \in \{1, \dots, d\}^{k-2}} \left\| \sum_{g,h=1}^d X^{(g,h,\tilde{a}')} b(e_g, e_h) e_{\tilde{a}'_1} \otimes \cdots \otimes e_{\tilde{a}'_{k-2}} \right\|_\alpha^{\bullet,2} \\ &= \frac{k^2(k-1)^2}{4} \sum_{\tilde{a}' \in \{1, \dots, d\}^{k-2}} \left| \sum_{g,h=1}^d X^{(g,h,\tilde{a}')} b(e_g, e_h) \right|^2 (k-2)! \\ &\leq \frac{k(k-1)k!}{4} \sum_{\tilde{a}' \in \{1, \dots, d\}^{k-2}} \left( \sum_{g,h=1}^d |X^{(g,h,\tilde{a}')}| |b(e_g, e_h)| \right)^2 \\ &\stackrel{\text{CS}}{\leq} \frac{k(k-1)k!}{4} \sum_{\tilde{a}' \in \{1, \dots, d\}^{k-2}} \left( \sum_{g,h=1}^d |X^{(g,h,\tilde{a}')}|^2 \right) \left( \sum_{g,h=1}^d |b(e_g, e_h)|^2 \right) \\ &\leq \frac{k(k-1)k!}{4} \sum_{a' \in \{1, \dots, d\}^k} |X^{a'}|^2 \\ &= \frac{k(k-1)}{4} \|X\|_\alpha^{\bullet,2}. \end{aligned}$$

Using this we get

$$\begin{aligned} \|(\Delta_b)^t X\|_\alpha^{\bullet,2} &= \sum_{k=2t}^{\infty} \|(\Delta_b)^t \langle X \rangle_k\|_\alpha^{\bullet,2} \\ &\leq \sum_{k=2t}^{\infty} \binom{k}{2t} \frac{(2t)!}{4^t} \|\langle X \rangle_k\|_\alpha^{\bullet,2} \\ &\leq \frac{(2t)!}{4^t} \sum_{k=2t}^{\infty} \frac{1}{r^k} \|\langle X \rangle_k\|_{2r\alpha}^{\bullet,2} \\ &\leq \frac{(2t)!}{(2r)^{2t}} \|X\|_{2r\alpha}^{\bullet,2} \end{aligned}$$

for arbitrary  $X \in \mathcal{T}^\bullet(V)$  and  $t \in \mathbb{N}$ . Finally, the estimate (3.6) also holds in the case  $t = 0$ .  $\square$

**Theorem 3.10** *Let  $b$  be a symmetric bilinear form on  $V$ , then the linear operator  $e^{\Delta_b} = \sum_{t=0}^{\infty} \frac{1}{t!} (\Delta_b)^t$  as well as its restriction to  $\mathcal{S}^\bullet(V)$  are continuous if and only if  $b$  is of Hilbert-Schmidt type. In this case*

$$e^{\Delta_b}(X \star_\Lambda Y) = (e^{\Delta_b} X) \star_{\Lambda+b} (e^{\Delta_b} Y) \quad (3.7)$$

*holds for all  $X, Y \in \mathcal{S}^\bullet(V)$  and all continuous bilinear forms  $\Lambda$  on  $V$ . Hence  $e^{\Delta_b}$  describes an isomorphism of the locally convex algebras  $(\mathcal{S}^\bullet(V), \star_\Lambda)$  and  $(\mathcal{S}^\bullet(V), \star_{\Lambda+b})$ . Moreover, for fixed  $X \in \mathcal{S}^\bullet(V)^{\text{cpl}}$ , the series  $e^{z\Delta_b} X$  converges absolutely and locally uniformly in  $z \in \mathbb{C}$  and thus depends holomorphically on  $z$ .*

PROOF: As  $|\Delta_b X| \leq \|e^{\Delta_b} X\|_\alpha^\bullet$  holds for all  $\|\cdot\|_\alpha \in \mathcal{P}_V$  and all  $X \in \mathcal{S}^2(V)$ , it follows from Proposition 3.8 that continuity of the restriction of  $e^{\Delta_b}$  to  $\mathcal{S}^\bullet(V)$  implies that  $b$  is of Hilbert-Schmidt type. Conversely, for all  $X \in \mathcal{T}^\bullet(V)$ , all  $\alpha \in \mathcal{P}_{V,b,HS}$ , and  $r > 1$ , the estimate

$$\|e^{z\Delta_b} X\|_\alpha \leq \sum_{t=0}^{\infty} \frac{1}{t!} \|(z\Delta_b)^t(X)\|_\alpha \leq \sum_{t=0}^{\infty} \frac{|z|^t}{(4r)^t} \binom{2t}{t}^{\frac{1}{2}} \|X\|_{4r\alpha}^\bullet \leq \sum_{t=0}^{\infty} \frac{1}{2^t} \|X\|_{4r\alpha}^\bullet = 2\|X\|_{4r\alpha}^\bullet$$

holds for all  $z \in \mathbb{C}$  with  $|z| \leq r$  due to the previous Proposition 3.9 if  $b$  is of Hilbert-Schmidt type, which proves the continuity of  $e^{z\Delta_b}$  for all  $z \in \mathbb{C}$  as well as the absolute and locally uniform convergence of the series  $e^{z\Delta_b} X$ . The algebraic relation (3.7) is well-known, see e.g. [23, Prop. 2.18]. Finally, as  $e^{\Delta_b}$  is invertible with inverse  $e^{-\Delta_b}$ , and because  $\Delta_b$  and thus  $e^{\Delta_b}$  map symmetric tensors to symmetric ones, we conclude that the restriction of  $e^{\Delta_b}$  to  $\mathcal{S}^\bullet(V)$  is an isomorphism of the locally convex algebras  $(\mathcal{S}^\bullet(V), \star_\Lambda)$  and  $(\mathcal{S}^\bullet(V), \star_{\Lambda+b})$ .  $\square$

### 3.3 Gel'fand Transformation

We are now going to construct an isomorphism of the undeformed  $\ast$ -algebra  $(\mathcal{S}^\bullet(V), \vee, \ast)$  to a  $\ast$ -algebra of smooth functions by a construction similar to the Gel'fand transformation of commutative  $C^\ast$ -algebras.

Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$ . We write  $V_h$  for the real linear subspace of  $V$  consisting of Hermitian elements, i.e.

$$V_h := \{v \in V \mid \bar{v} = v\}. \quad (3.8)$$

The inner products compatible with the involution are denoted by

$$\mathcal{I}_{V,h} := \{\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_V \mid \overline{\langle v \mid w \rangle_\alpha} = \langle \bar{v} \mid \bar{w} \rangle_\alpha \text{ for all } v, w \in V\}. \quad (3.9)$$

Moreover, we write  $V'$  for the topological dual space of  $V$  and  $V'_h$  again for the real linear subspace of  $V'$  consisting of Hermitian elements, i.e.

$$V'_h := \{\rho \in V' \mid \overline{\rho(v)} = \rho(\bar{v}) \text{ for all } v \in V\}. \quad (3.10)$$

Finally, recall that a subset  $B \subseteq V'_h$  is *bounded* (with respect to the equicontinuous bornology) if there exists a  $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  such that  $|\rho(v)| \leq \|v\|_\alpha$  holds for all  $v \in V$  and all  $\rho \in B$ . This also gives a notion of boundedness of functions from or to  $V'_h$ : A (multi-)linear function is bounded if it maps bounded sets to bounded ones.

Note that one can identify  $V'_h$  with the topological dual of  $V_h$  and  $\mathcal{I}_{V,h}$  with the set of continuous positive bilinear forms on  $V_h$ . Moreover,  $\mathcal{I}_{V,h}$  is cofinal in  $\mathcal{I}_V$ : every  $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_V$  is dominated by  $V^2 \ni (v, w) \mapsto \langle v \mid w \rangle_\alpha + \overline{\langle v \mid w \rangle_\alpha} \in \mathcal{I}_{V,h}$ .

**Definition 3.11** Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$  and  $\rho \in V'_h$ , then we define the derivative in direction of  $\rho$  as the linear map  $D_\rho: \mathcal{T}^\bullet(V) \rightarrow \mathcal{T}^{\bullet-1}(V)$  by

$$x_1 \otimes \cdots \otimes x_k \mapsto D_\rho(x_1 \otimes \cdots \otimes x_k) := k\rho(x_k)x_1 \otimes \cdots \otimes x_{k-1} \quad (3.11)$$

for all  $k \in \mathbb{N}$  and all  $x \in V^k$ . Next, we define the translation by  $\rho$  as the linear map

$$\tau_\rho^* := \sum_{t=0}^{\infty} \frac{1}{t!} (D_\rho)^t: \mathcal{T}^\bullet(V) \rightarrow \mathcal{T}^\bullet(V), \quad (3.12)$$

and the evaluation at  $\rho$  by

$$\delta_\rho := \langle \cdot \rangle_0 \circ \tau_\rho^*: \mathcal{T}^\bullet(V) \rightarrow \mathbb{C}. \quad (3.13)$$

Finally, for  $k \in \mathbb{N}$  and  $\rho_1, \dots, \rho_k \in V'_h$  we set  $D_{\rho_1, \dots, \rho_k}^{(k)} := D_{\rho_1} \cdots D_{\rho_k}: \mathcal{T}^\bullet(V) \rightarrow \mathcal{T}^{\bullet-k}(V)$ .

Note that  $\tau_\rho^*$  is well-defined because for every  $X \in \mathcal{T}^\bullet(V)$  only finitely many terms contribute to the infinite series  $\tau_\rho^* X = \sum_{t=0}^{\infty} \frac{1}{t!} (D_\rho)^t(X)$ . Note also that  $D_\rho$  and consequently also  $\tau_\rho^*$  can be restricted to endomorphisms of  $\mathcal{S}^\bullet(V)$ . Moreover, this restriction of  $D_\rho$  is a  $*$ -derivation of all the  $*$ -algebras  $(\mathcal{S}^\bullet(V), \star_\Lambda, *)$  for all continuous Hermitian bilinear forms  $\Lambda$  on  $V$  (see [23, Lem. 2.13, iii], the compatibility with the  $*$ -involution is clear), so that  $\tau_\rho^*$  turns out to be a unital  $*$ -automorphism of these  $*$ -algebras.

**Lemma 3.12** Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$  and  $\rho, \sigma \in V'_h$ . Then

$$(D_\rho D_\sigma - D_\sigma D_\rho)(X) = (\tau_\rho^* D_\sigma - D_\sigma \tau_\rho^*)(X) = (\tau_\rho^* \tau_\sigma^* - \tau_\sigma^* \tau_\rho^*)(X) = 0 \quad (3.14)$$

holds for all  $X \in \mathcal{S}^\bullet(V)$ .

PROOF: It is sufficient to show that  $(D_\rho D_\sigma - D_\sigma D_\rho)(X) = 0$  for all  $X \in \mathcal{S}^\bullet(V)$ , which clearly holds if  $X$  is a homogeneous factorizing symmetric tensor and so holds for all  $X \in \mathcal{S}^\bullet(V)$  by linearity.  $\square$

**Lemma 3.13** Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$  and  $\rho \in V'_h$ . Then  $D_\rho$ ,  $\tau_\rho^*$  and  $\delta_\rho$  are all continuous. Moreover, if  $\|\cdot\|_\alpha \in \mathcal{P}_V$  fulfills  $|\rho(v)| \leq \|v\|_\alpha$ , then the estimates

$$\|(D_\rho)^t X\|_\alpha^\bullet \leq \sqrt{t!} \|X\|_{2\alpha}^\bullet \quad (3.15)$$

and

$$\|\tau_\rho^*(X)\|_\alpha^\bullet \leq \sum_{t'=0}^{\infty} \frac{1}{t'!} \|(D_\rho)^{t'} X\|_\alpha^\bullet \leq \frac{2}{\sqrt{2}-1} \|X\|_{2\alpha}^\bullet \quad (3.16)$$

hold for all  $X \in \mathcal{T}^\bullet(V)$  and all  $t \in \mathbb{N}_0$ .

PROOF: Let  $\|\cdot\|_\alpha \in \mathcal{P}_V$  be given such that  $|\rho(v)| \leq \|v\|_\alpha$  holds for all  $v \in V$ . For all  $d \in \mathbb{N}_0$  and all  $\langle \cdot | \cdot \rangle_\alpha$ -orthonormal  $e \in V^d$  we then get

$$\sum_{i=1}^d |\rho(e_i)|^2 = \rho \left( \sum_{i=1}^d e_i \overline{\rho(e_i)} \right) \leq \left\| \sum_{i=1}^d e_i \overline{\rho(e_i)} \right\|_\alpha = \left( \sum_{i=1}^d |\rho(e_i)|^2 \right)^{\frac{1}{2}},$$

hence  $\sum_{i=1}^d |\rho(e_i)|^2 \leq 1$ . Given  $k \in \mathbb{N}$  and a tensor  $X \in \mathcal{T}^k(V)$ , then we construct  $X_0 = \sum_{a \in A} x_{a,1} \otimes \cdots \otimes x_{a,k}$  and  $\tilde{X} = \sum_{a' \in \{1, \dots, d\}^k} X^{a'} e_{a'_1} \otimes \cdots \otimes e_{a'_k}$  like in Lemma 2.3. Then we have  $\|D_\rho X_0\|_\alpha^\bullet = 0$  because

$$\|D_\rho(x_{a,1} \otimes \cdots \otimes x_{a,k})\|_\alpha^\bullet = k|\rho(x_{a,k})| \|x_{a,1} \otimes \cdots \otimes x_{a,k-1}\|_\alpha^\bullet \leq k\sqrt{(k-1)!} \prod_{m=1}^k \|x_{a,m}\|_\alpha = 0$$



holds for all  $a \in A$ . Consequently  $\|D_\rho X\|_\alpha^\bullet \leq \|D_\rho \tilde{X}\|_\alpha^\bullet$  and we get

$$\begin{aligned}
\|D_\rho X\|_\alpha^{\bullet,2} &\leq \|D_\rho \tilde{X}\|_\alpha^{\bullet,2} = \left\| \sum_{a' \in \{1, \dots, d\}^k} X^{a'} D_\rho(e_{a'_1} \otimes \dots \otimes e_{a'_k}) \right\|_\alpha^{\bullet,2} \\
&= k^2 \sum_{\tilde{a}' \in \{1, \dots, d\}^{k-1}} \left\| \sum_{g=1}^d X^{(\tilde{a}', g)} \rho(e_g) e_{\tilde{a}'_1} \otimes \dots \otimes e_{\tilde{a}'_{k-1}} \right\|_\alpha^{\bullet,2} \\
&\leq k^2 (k-1)! \sum_{\tilde{a}' \in \{1, \dots, d\}^{k-1}} \left( \sum_{g=1}^d |X^{(\tilde{a}', g)}| |\rho(e_g)| \right)^2 \\
&\stackrel{\text{CS}}{\leq} k^2 (k-1)! \sum_{\tilde{a}' \in \{1, \dots, d\}^{k-1}} \left( \sum_{g=1}^d |X^{(\tilde{a}', g)}|^2 \right) \left( \sum_{g=1}^d |\rho(e_g)|^2 \right) \\
&\leq k^2 (k-1)! \sum_{a' \in \{1, \dots, d\}^k} |X^{a'}|^2 \\
&= k \|X\|_\alpha^{\bullet,2}.
\end{aligned}$$

Using this we can derive the estimate (3.15), which also proves the continuity of  $D_\rho$ : If  $t = 0$ , then this is clearly fulfilled. Otherwise, let  $X \in \mathcal{T}^\bullet(V)$  be given, then

$$\|(D_\rho)^t X\|_\alpha^{\bullet,2} = \sum_{k=t}^{\infty} \|(D_\rho)^t \langle X \rangle_k\|_\alpha^{\bullet,2} \leq t! \sum_{k=t}^{\infty} \binom{k}{t} \|\langle X \rangle_k\|_\alpha^{\bullet,2} \leq t! \sum_{k=t}^{\infty} \|\langle X \rangle_k\|_{2\alpha}^{\bullet,2} \leq t! \|X\|_{2\alpha}^{\bullet,2}.$$

From this we can now also deduce the estimate (3.16), which then shows continuity of  $\tau_\rho^*$  and of  $\delta_\rho = \langle \cdot \rangle_0 \circ \tau_\rho^*$ : The first inequality is just the triangle inequality and for the second we use that  $t! \geq 2^{t-1}$  for all  $t \in \mathbb{N}_0$ , so

$$\sum_{t=0}^{\infty} \frac{1}{t!} \|(D_\rho)^t X\|_\alpha^\bullet \leq \sum_{t=0}^{\infty} \frac{1}{\sqrt{t!}} \|X\|_{2\alpha}^\bullet \leq \sqrt{2} \sum_{t=0}^{\infty} \frac{1}{\sqrt{2}^t} \|X\|_{2\alpha}^\bullet \leq \frac{2}{\sqrt{2}-1} \|X\|_{2\alpha}^\bullet. \quad \square$$

**Proposition 3.14** *Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$ , then the set of all continuous unital  $*$ -homomorphisms from  $(\mathcal{S}^\bullet(V)^{\text{cpl}}, \vee, *)$  to  $\mathbb{C}$  is  $\{\delta_\rho \mid \rho \in V'_h\}$  (strictly speaking, the continuous extensions to  $\mathcal{S}^\bullet(V)^{\text{cpl}}$  of the restrictions of  $\delta_\rho$  to  $\mathcal{S}^\bullet(V)$ ).*

PROOF: On the one hand, every such  $\delta_\rho$  is a continuous unital  $*$ -homomorphism, because  $\langle \cdot \rangle_0$  and  $\tau_\rho^*$  are. On the other hand, if  $\phi: (\mathcal{S}^\bullet(V)^{\text{cpl}}, \vee, *) \rightarrow \mathbb{C}$  is a continuous unital  $*$ -homomorphism, then  $V \ni v \mapsto \rho(v) := \phi(v) \in \mathbb{C}$  is an element of  $V'_h$  and fulfills  $\delta_\rho = \phi$  because the unital  $*$ -algebra  $(\mathcal{S}^\bullet(V), \vee, *)$  is generated by  $V$  and because  $\mathcal{S}^\bullet(V)$  is dense in its completion.  $\square$

Let  $\Phi := \{\delta_\rho \mid \rho \in V'_h\}$  be the set of all continuous unital  $*$ -homomorphisms from  $(\mathcal{S}^\bullet(V)^{\text{cpl}}, \vee, *)$  to  $\mathbb{C}$  and  $\mathbb{C}^\Phi$  the unital  $*$ -algebra of all functions from  $\Phi$  to  $\mathbb{C}$  with the pointwise operations, then the Gel'fand-transformation is usually defined as the unital  $*$ -homomorphism  $\sim: (\mathcal{S}^\bullet(V)^{\text{cpl}}, \vee, *) \rightarrow \mathbb{C}^\Phi$ ,  $X \mapsto \tilde{X}$  with  $\tilde{X}(\phi) := \phi(X)$  for all  $\phi \in \Phi$ . This is a natural way to transform an abstract commutative unital locally convex  $*$ -algebras to a  $*$ -algebra of complex-valued functions. For our purposes, however, it will be more convenient to identify  $\Phi$  with  $V'_h$  like in the previous Proposition 3.14:

**Definition 3.15** *Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$  and  $X \in \mathcal{S}^\bullet(V)^{\text{cpl}}$ , then we define the function  $\hat{X}: V'_h \rightarrow \mathbb{C}$  by*

$$\rho \mapsto \hat{X}(\rho) := \delta_\rho(X). \quad (3.17)$$

In the following we will show that this construction yields an isomorphism between  $(\mathcal{S}^\bullet(V)^{\text{cpl}}, \vee, *)$  and a unital  $*$ -algebra of certain functions on  $V'_h$ :

**Definition 3.16** Let  $f: V'_h \rightarrow \mathbb{C}$  be a function. For  $\rho, \sigma \in V'_h$  we denote by

$$(\widehat{D}_\rho f)(\sigma) := \left. \frac{d}{dt} \right|_{t=0} f(\sigma + t\rho) \quad (3.18)$$

(if it exists) the directional derivative of  $f$  at  $\sigma$  in direction  $\rho$ . If the directional derivative of  $f$  in direction  $\rho$  exists at all  $\sigma \in V'_h$ , then we denote by  $\widehat{D}_\rho f: V'_h \rightarrow \mathbb{C}$  the function  $\sigma \mapsto (\widehat{D}_\rho f)(\sigma)$ . In this case we can also examine directional derivatives of  $\widehat{D}_\rho f$  and define the iterated directional derivative

$$\widehat{D}_\rho^{(k)} f := \widehat{D}_{\rho_1} \cdots \widehat{D}_{\rho_k} f \quad (3.19)$$

(if it exists) for  $k \in \mathbb{N}$  and  $\rho \in (V'_h)^k$ . For  $k = 0$  we define  $\widehat{D}^{(0)} f := f$ . Moreover, we say that  $f$  is smooth if all iterated directional derivatives  $\widehat{D}_\rho^{(k)} f$  exist for all  $k \in \mathbb{N}_0$  and all  $\rho \in (V'_h)^k$  and describe a bounded symmetric multilinear form  $(V'_h)^k \ni \rho \mapsto (\widehat{D}_\rho^{(k)} f)(\sigma) \in \mathbb{C}$  for all  $\sigma \in V'_h$ . Finally, we write  $\mathcal{C}^\infty(V'_h)$  for the unital  $*$ -algebra of all smooth functions on  $V'_h$ .

Note that this notion of smoothness is rather weak, we do not even demand that a smooth function is continuous (we did not even endow  $V'_h$  with a topology). For example, every bounded linear functional on  $V'_h$  is smooth.

**Proposition 3.17** Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$  and  $X \in \mathcal{S}^\bullet(V)^{\text{cpl}}$ . Then  $\widehat{X}: V'_h \rightarrow \mathbb{C}$  is smooth and

$$\widehat{D}_\rho^{(k)} \widehat{X} = \widehat{D_\rho^{(k)} X} \quad (3.20)$$

holds for all  $k \in \mathbb{N}_0$  and all  $\rho \in (V'_h)^k$ .

PROOF: Let  $X \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  be given. As the exponential series  $\tau_{t\rho}^*(X)$  is absolutely convergent by Lemma 3.13, it follows that  $\left. \frac{d}{dt} \right|_{t=0} \tau_{t\rho}^*(X) = D_\rho(X)$  for all  $\rho \in V'_h$  and so we conclude that

$$(\widehat{D}_\rho \widehat{X})(\sigma) = \left. \frac{d}{dt} \right|_{t=0} \delta_{\sigma+t\rho}(X) = \left\langle \tau_\sigma^* \left( \left. \frac{d}{dt} \right|_{t=0} \tau_{t\rho}^*(X) \right) \right\rangle_0 = \langle \tau_\sigma^*(D_\rho(X)) \rangle_0 = \widehat{D_\rho(X)}(\sigma)$$

holds for all  $\rho, \sigma \in V'_h$ , which proves (3.20) in the case  $k = 1$ . We see that  $\widehat{D}_\rho$  for all  $\rho \in V'_h$  is an endomorphism of the vector space  $\{\widehat{X} \mid X \in \mathcal{S}^\bullet(V)^{\text{cpl}}\}$ , so all iterated directional derivatives of such an  $\widehat{X}$  exist. By induction it is now easy to see that (3.20) holds for arbitrary  $k \in \mathbb{N}_0$ . Moreover,  $D_\rho D_{\rho'} X = D_{\rho'} D_\rho X$  holds for all  $\rho, \rho' \in V'_h$  and all  $X \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  by Lemmas 3.12 and 3.13. Together with (3.20) this shows that directional derivatives on  $\widehat{X}$  commute. Finally, the multilinear form  $(V'_h)^k \ni \rho \mapsto (\widehat{D}_\rho^{(k)} \widehat{X})(\sigma) \in \mathbb{C}$  is bounded for all  $\sigma \in V'_h$ : It is sufficient to show this for  $\sigma = 0$ , because  $\tau_\sigma^*$  is a continuous automorphism of  $\mathcal{S}^\bullet(V)$  and commutes with  $D_\rho^{(k)}$ . If  $\rho \in (V'_h)^k$  fulfills  $|\rho_i(v)| \leq \|v\|_\alpha$  for all  $i \in \{1, \dots, k\}$ , all  $v \in V$  and one  $\|\cdot\|_\alpha \in \mathcal{P}_V$ , then we have  $\|D_{\rho_1} \cdots D_{\rho_k} X\|_\alpha^\bullet \leq \|X\|_{2^k \alpha}$  due to Lemma 3.13, which is an upper bound of  $(\widehat{D}_\rho^{(k)} \widehat{X})(0)$ .  $\square$

Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$  and let  $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  be given, then the degeneracy space of the inner product  $\langle \cdot \mid \cdot \rangle_\alpha$  is

$$\ker_h \|\cdot\|_\alpha := \{v \in V_h \mid \|v\|_\alpha = 0\}. \quad (3.21)$$

Thus we get a well-defined non-degenerate positive bilinear form on the real vector space  $V_h/\ker_h\|\cdot\|_\alpha$ . We write  $V_{h,\alpha}^{\text{cpl}}$  for the completion of this space to a real Hilbert space with inner product  $\langle \cdot | \cdot \rangle_\alpha$  and define the linear map  $\cdot^{\flat_\alpha}$  from  $V_{h,\alpha}^{\text{cpl}}$  to  $V'_h$  as

$$v^{\flat_\alpha}(w) := \langle v | w \rangle_\alpha \quad (3.22)$$

for all  $v \in V_{h,\alpha}^{\text{cpl}}$  and all  $w \in V$ . Note that  $\cdot^{\flat_\alpha}: V_{h,\alpha}^{\text{cpl}} \rightarrow V'_h$  is a bounded linear map due to the Cauchy-Schwarz inequality. Analogously, we define

$$\ker\|\cdot\|_\alpha^\bullet := \{X \in \mathcal{T}^\bullet(V) \mid \|X\|_\alpha^\bullet = 0\}, \quad (3.23)$$

and denote by  $\mathcal{T}^\bullet(V)_\alpha^{\text{cpl}}$  the completion of the complex vector space  $\mathcal{T}_{\text{alg}}^\bullet(V)/\ker\|\cdot\|_\alpha^\bullet$  to a complex Hilbert space with inner product  $\langle \cdot | \cdot \rangle_\alpha^\bullet$ . Then  $\mathcal{S}^\bullet(V)_\alpha^{\text{cpl}}$  becomes the linear subspace of (equivalence classes of) symmetric tensors, which is closed because  $\mathcal{S}^\bullet$  extends to a continuous endomorphism of  $\mathcal{T}^\bullet(V)_\alpha^{\text{cpl}}$  by Proposition 2.7.

Moreover, for all  $\langle \cdot | \cdot \rangle_\alpha, \langle \cdot | \cdot \rangle_\beta \in \mathcal{I}_{V,h}$  with  $\langle \cdot | \cdot \rangle_\beta \leq \langle \cdot | \cdot \rangle_\alpha$ , the linear map  $\text{id}_{\mathcal{T}^\bullet(V)}: \mathcal{T}^\bullet(V) \rightarrow \mathcal{T}^\bullet(V)$  extends to continuous linear maps  $\iota_{\infty\alpha}: \mathcal{T}^\bullet(V)^{\text{cpl}} \rightarrow \mathcal{T}^\bullet(V)_\alpha^{\text{cpl}}$  and  $\iota_{\alpha\beta}: \mathcal{T}^\bullet(V)_\alpha^{\text{cpl}} \rightarrow \mathcal{T}^\bullet(V)_\beta^{\text{cpl}}$ , such that  $\iota_{\alpha\beta} \circ \iota_{\infty\alpha} = \iota_{\infty\beta}$  and  $\iota_{\beta\gamma} \circ \iota_{\alpha\beta} = \iota_{\alpha\gamma}$  hold for all  $\langle \cdot | \cdot \rangle_\alpha, \langle \cdot | \cdot \rangle_\beta, \langle \cdot | \cdot \rangle_\gamma \in \mathcal{I}_{V,h}$  with  $\langle \cdot | \cdot \rangle_\gamma \leq \langle \cdot | \cdot \rangle_\beta \leq \langle \cdot | \cdot \rangle_\alpha$ . This way,  $\mathcal{T}^\bullet(V)^{\text{cpl}}$  is realized as the projective limit of the Hilbert spaces  $\mathcal{T}^\bullet(V)_\alpha^{\text{cpl}}$  and similarly,  $\mathcal{S}^\bullet(V)^{\text{cpl}}$  as the projective limit of the closed linear subspaces  $\mathcal{S}^\bullet(V)_\alpha^{\text{cpl}}$ .

**Lemma 3.18** *Let  $\overline{\cdot}$  be a continuous antilinear involution on  $V$  and  $f \in \mathcal{C}^\infty(V'_h)$ . Given  $\rho \in V'_h$  and  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  such that  $|\rho(v)| \leq \|v\|_\alpha$  holds for all  $v \in V$ , then*

$$\widehat{D}_\rho f = \sum_{i \in I} \rho(e_i) \widehat{D}_{e_i^{\flat_\alpha}} f \quad (3.24)$$

*holds for every Hilbert basis  $e \in (V_{h,\alpha}^{\text{cpl}})^I$  of  $V_{h,\alpha}^{\text{cpl}}$  indexed by a set  $I$ .*

PROOF: As  $f$  is smooth, the function  $V'_h \ni \sigma \mapsto \widehat{D}_\sigma f \in \mathbb{C}$  is bounded, which implies that its restriction to the dual space of  $V_{h,\alpha}^{\text{cpl}}$  is continuous with respect to the Hilbert space topology on (the dual of)  $V_{h,\alpha}^{\text{cpl}}$ . As  $\rho = \sum_{i \in I} e_i^{\flat_\alpha} \rho(e_i)$  with respect to this topology, it follows that  $\widehat{D}_\rho f = \sum_{i \in I} \rho(e_i) \widehat{D}_{e_i^{\flat_\alpha}} f$ .  $\square$

**Definition 3.19 (Hilbert-Schmidt type functions)** *Let  $\overline{\cdot}$  be a continuous antilinear involution on  $V$ . We say that a function  $f: V'_h \rightarrow \mathbb{C}$  is analytic of Hilbert-Schmidt type, if it is smooth and additionally fulfills the condition that for all  $\sigma, \sigma' \in V'_h$  and all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  there exists a  $C_{\sigma,\sigma',\alpha} \in \mathbb{R}$  such that*

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \left| (\widehat{D}_{(e_{i_1}^{\flat_\alpha}, \dots, e_{i_k}^{\flat_\alpha})}^{(k)} f)(\xi) \right|^2 \leq C_{\sigma,\sigma',\alpha} \quad (3.25)$$

*holds for one Hilbert base  $e \in (V_{h,\alpha}^{\text{cpl}})^I$  of  $V_{h,\alpha}^{\text{cpl}}$  indexed by a set  $I$  and every  $\xi$  from the line-segment between  $\sigma$  and  $\sigma'$ , i.e. every  $\xi = \lambda\sigma + (1-\lambda)\sigma'$  with  $\lambda \in [0, 1]$ . We write  $\mathcal{C}^{\omega_{HS}}(V'_h)$  for the set of all complex functions on  $V'_h$  that are analytic of Hilbert-Schmidt type.*

Here and elsewhere a sum over an uncountable Hilbert basis is understood in the usual sense: only countably many terms in the sum are non-zero.

This definition is independent of the choice of the Hilbert basis due to Lemma 3.18 and  $\mathcal{C}^{\omega_{HS}}(V'_h)$  is a complex vector space. It is not too hard to check that  $\mathcal{C}^{\omega_{HS}}(V'_h)$  is even a unital  $\ast$ -subalgebra of  $\mathcal{C}^\infty(V'_h)$ . However, we will indirectly prove this later on. Calling the functions in  $\mathcal{C}^{\omega_{HS}}(V'_h)$  *analytic* is justified thanks to the following statement:

**Proposition 3.20** *Let  $\overline{\cdot}$  be a continuous antilinear involution on  $V$  and  $f: V'_h \rightarrow \mathbb{C}$  analytic of Hilbert-Schmidt type with  $(\widehat{D}_\rho^{(k)} f)(0) = 0$  for all  $k \in \mathbb{N}_0$  and all  $\rho \in (V'_h)^k$ . Then  $f = 0$ .*

PROOF: Given  $\sigma \in V'_h$ , then define the smooth function  $g: \mathbb{R} \rightarrow \mathbb{C}$  by  $t \mapsto g(t) := f(t\sigma)$ . We write  $g^{(k)}(t)$  for the  $k$ -th derivative of  $g$  at  $t$ . Then there exists a  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  that fulfills  $|\sigma(v)| \leq \|v\|_\alpha$  for all  $v \in V$ , and consequently  $\sigma = \nu e^{b_\alpha}$  with a normalized  $e \in V_{h,\alpha}^{\text{cpl}}$  and  $\nu \in [0, 1]$  by the Fréchet-Riesz theorem. Therefore,

$$\left( \sum_{k=0}^{\infty} \frac{1}{k!} |g^{(k)}(t)| \right)^2 \stackrel{\text{CS}}{\leq} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} |g^{(\ell)}(t)|^2 \leq e \sum_{\ell=0}^{\infty} \frac{\nu^{2\ell}}{\ell!} \left| (\widehat{D}_{(e^{b_\alpha}, \dots, e^{b_\alpha})}^{(\ell)} f)(t\sigma) \right|^2 \leq e C_{-2\sigma, 2\sigma, \alpha}$$

holds for all  $t \in [-2, 2]$  with a constant  $C_{-2\sigma, 2\sigma, \alpha} \in \mathbb{R}$ , which shows that  $g$  is an analytic function on  $] -2, 2[$ . As  $g^{(k)}(0) = 0$  for all  $k \in \mathbb{N}_0$  this implies  $f(\sigma) = g(1) = 0$ .  $\square$

Note that one can derive even better estimates for the derivatives of  $g$ . This shows that condition (3.25) is even stronger than just analyticity.

**Definition 3.21** *Let  $\overline{\cdot}$  be a continuous antilinear involution on  $V$  and let  $f, g: V'_h \rightarrow \mathbb{C}$  be analytic of Hilbert-Schmidt type as well as  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$ . Because of the estimate (3.25) we can define a function  $\langle\langle f | g \rangle\rangle_\alpha^\bullet: V'_h \rightarrow \mathbb{C}$  by*

$$\rho \mapsto \langle\langle f | g \rangle\rangle_\alpha^\bullet(\rho) := \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \overline{(\widehat{D}_{e_i^{b_\alpha}}^{(k)} f)(\rho)} (\widehat{D}_{e_i^{b_\alpha}}^{(k)} g)(\rho), \quad (3.26)$$

where  $e \in (V_{h,\alpha}^{\text{cpl}})^I$  is an arbitrary Hilbert base of  $V_{h,\alpha}^{\text{cpl}}$  indexed by a set  $I$ .

Note that  $\langle\langle f | g \rangle\rangle_\alpha^\bullet$  does not depend on the choice of this Hilbert base due to Lemma 3.18. Essentially,  $\langle\langle f | g \rangle\rangle_\alpha^\bullet(\rho)$  is a weighted  $\ell^2$ -inner product (yet not necessarily positive-definite) of all partial derivatives of  $f$  and  $g$  at  $\rho$  in directions described by (the dual of) a  $\langle \cdot | \cdot \rangle_\alpha$ -Hilbert base. Note that the analyticity condition (3.25) for a function  $f$  is equivalent to demanding that  $\langle\langle f | f \rangle\rangle_\alpha^\bullet(\xi)$  exists for all  $\xi \in V'_h$  and all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  and is uniformly bounded on line segments in  $V'_h$ .

**Lemma 3.22** *Let  $\overline{\cdot}$  be a continuous antilinear involution on  $V$ . Let  $k \in \mathbb{N}$  and  $x \in (V_h)^k$  as well as  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  be given. Then*

$$(\widehat{D}_{x^{b_\alpha}}^{(k)} \widehat{Y})(0) = \langle D_{x^{b_\alpha}}^{(k)} Y \rangle_0 = \langle x_1 \otimes \dots \otimes x_k | Y \rangle_\alpha^\bullet \quad (3.27)$$

holds for all  $Y \in \mathcal{S}^\bullet(V)^{\text{cpl}}$ .

PROOF: The first identity is just Proposition 3.17, and for the second one it is sufficient to show that  $\langle D_{x^{b_\alpha}}^{(k)} Y \rangle_0 = \langle x_1 \otimes \dots \otimes x_k | Y \rangle_\alpha^\bullet$  holds for all factorizing tensors  $Y$  of degree  $k$ , because both sides of this equation vanish on homogeneous tensors of different degree and are linear and continuous in  $Y$  by Lemma 3.13. However, it is an immediate consequence of the definitions of  $D$ ,  $\cdot^{b_\alpha}$ , and  $\langle \cdot | \cdot \rangle_\alpha^\bullet$  that

$$\left\langle D_{(x_1^{b_\alpha}, \dots, x_k^{b_\alpha})}^{(k)} y_1 \otimes \dots \otimes y_k \right\rangle_0 = k! \prod_{m=1}^k \langle x_m | y_m \rangle_\alpha = \langle x_1 \otimes \dots \otimes x_k | y_1 \otimes \dots \otimes y_k \rangle_\alpha^\bullet$$

holds for all  $y_1, \dots, y_k \in V$ .  $\square$

**Proposition 3.23** *Let  $\overline{\cdot}$  be a continuous antilinear involution on  $V$ , then*

$$\langle\langle \widehat{X} | \widehat{Y} \rangle\rangle_\alpha^\bullet(\rho) = \langle \tau_\rho^* X | \tau_\rho^* Y \rangle_\alpha^\bullet = \widehat{X^* \star_{\Lambda_\alpha} Y}(\rho) \quad (3.28)$$

holds for all  $X, Y \in \mathcal{S}^\bullet(V)^{\text{cpl}}$ , all  $\rho \in V'_h$ , and all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$ , where  $\Lambda_\alpha: V \times V \rightarrow \mathbb{C}$  is the continuous bilinear form defined by  $\Lambda_\alpha(v, w) := \langle \overline{v} | w \rangle_\alpha$ .

PROOF: Let  $X, Y \in \mathcal{S}^\bullet(V)^{\text{cpl}}$ ,  $\rho \in V'_h$  and  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  be given. Let  $e \in (V_{h,\alpha}^{\text{cpl}})^I$  be a Hilbert base of  $V_{h,\alpha}^{\text{cpl}}$  indexed by a set  $I$ . Then

$$\begin{aligned} \langle \widehat{X} | \widehat{Y} \rangle_\alpha^\bullet(\rho) &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \overline{\left( \widehat{D}_{e_i^{\flat_\alpha}}^{(k)} \widehat{X} \right)(\rho)} \left( \widehat{D}_{e_i^{\flat_\alpha}}^{(k)} \widehat{Y} \right)(\rho) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \overline{\left\langle D_{e_i^{\flat_\alpha}}^{(k)} \tau_\rho^* X \right\rangle_0} \left\langle D_{e_i^{\flat_\alpha}}^{(k)} \tau_\rho^* Y \right\rangle_0 \\ &= \sum_{k=0}^{\infty} \sum_{i \in I^k} \frac{1}{k!} \langle \tau_\rho^* X | e_{i_1} \otimes \cdots \otimes e_{i_k} \rangle_\alpha^\bullet \langle e_{i_1} \otimes \cdots \otimes e_{i_k} | \tau_\rho^* Y \rangle_\alpha^\bullet \\ &= \langle \tau_\rho^* X | \tau_\rho^* Y \rangle_\alpha^\bullet \end{aligned}$$

holds by Proposition 3.17 and Lemma 3.12 as well as the previous Lemma 3.22 and the fact that the tensors  $(k!)^{1/2} e_{i_1} \otimes \cdots \otimes e_{i_k}$  for all  $k \in \mathbb{N}_0$  and  $i \in I^k$  form a Hilbert base of  $\mathcal{T}^\bullet(V)_\alpha^{\text{cpl}}$ . The second identity is a direct consequence of Theorem 3.5 because  $\tau_\rho^*$  is a unital  $*$ -automorphism of  $\star_{\Lambda_\alpha}$ . Indeed, we have

$$\langle \tau_\rho^* X | \tau_\rho^* Y \rangle_\alpha^\bullet = \langle (\tau_\rho^* X)^* \star_{\Lambda_\alpha} (\tau_\rho^* Y) \rangle_0 = \langle \tau_\rho^* (X^* \star_{\Lambda_\alpha} Y) \rangle_0 = \widehat{X^* \star_{\Lambda_\alpha} Y}(\rho). \quad \square$$

**Corollary 3.24** *Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$  and  $X \in \mathcal{S}^\bullet(V)^{\text{cpl}}$ , then  $\widehat{X} \in \mathcal{C}^{\omega_{HS}}(V'_h)$ .*

PROOF: The function  $\widehat{X}$  is smooth by Proposition 3.17. By the previous Proposition 3.23, we have

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \left| \left( \widehat{D}_{e_i^{\flat_\alpha}}^{(k)} \widehat{X} \right)(\xi) \right|^2 = \langle \widehat{X} | \widehat{X} \rangle_\alpha^\bullet(\xi) = \widehat{X^* \star_{\Lambda_\alpha} X}(\xi)$$

for all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$ , which is finite and depends smoothly on  $\xi \in V'_h$  by Proposition 3.17 again. Therefore it is uniformly bounded on line segments.  $\square$

**Lemma 3.25** *Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$  and  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$ . For every  $f \in \mathcal{C}^{\omega_{HS}}(V'_h)$  there exists an  $X_f \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  that fulfills  $\langle f | f \rangle_\alpha^\bullet(0) = \langle \widehat{X}_f | \widehat{X}_f \rangle_\alpha^\bullet(0)$  and  $\langle f | \widehat{Y} \rangle_\alpha^\bullet(0) = \langle \widehat{X}_f | \widehat{Y} \rangle_\alpha^\bullet(0)$  for all  $Y \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  and all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$ .*

PROOF: For every  $\alpha \in \mathcal{I}_{V,h}$  construct  $X_{f,\alpha} \in \mathcal{S}^\bullet(V)_\alpha^{\text{cpl}}$  as

$$X_{f,\alpha} := \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} e_{i_1} \otimes \cdots \otimes e_{i_k} \left( \widehat{D}_{e_i^{\flat_\alpha}}^{(k)} f \right)(0) \in \mathcal{S}^\bullet(V)_\alpha^{\text{cpl}},$$

where  $e \in (V_{h,\alpha}^{\text{cpl}})^I$  is a Hilbert base of  $V_{h,\alpha}^{\text{cpl}}$  indexed by a set  $I$ . This infinite sum  $X_{f,\alpha}$  indeed lies in  $\mathcal{S}^\bullet(V)_\alpha^{\text{cpl}}$  and fulfills  $\langle X_{f,\alpha} | X_{f,\alpha} \rangle_\alpha^\bullet = \langle f | f \rangle_\alpha^\bullet(0)$ , because  $\left( \widehat{D}_{e_i^{\flat_\alpha}}^{(k)} f \right)(0)$  is invariant under permutations of the  $e_{i_1}, \dots, e_{i_k}$  due to the smoothness of  $f$  and because

$$\begin{aligned} \sum_{k,\ell=0}^{\infty} \sum_{i \in I^k, i' \in I^\ell} \frac{1}{k! \ell!} \left\langle e_{i_1} \otimes \cdots \otimes e_{i_k} \left( \widehat{D}_{e_i^{\flat_\alpha}}^{(k)} f \right)(0) \middle| e_{i'_1} \otimes \cdots \otimes e_{i'_\ell} \left( \widehat{D}_{e_{i'}^{\flat_\alpha}}^{(\ell)} f \right)(0) \right\rangle_\alpha^\bullet \\ = \sum_{k=0}^{\infty} \sum_{i \in I^k} \frac{1}{k!} \left| \left( \widehat{D}_{e_i^{\flat_\alpha}}^{(k)} f \right)(0) \right|^2 \end{aligned}$$

$$= \langle\langle f | f \rangle\rangle_\alpha^\bullet(0).$$

Moreover, for all  $Y \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  the identity

$$\begin{aligned} \langle\langle f | \hat{Y} \rangle\rangle_\alpha^\bullet(0) &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \overline{\left( \hat{D}_{e_i^{\flat_\alpha}}^{(k)} f \right)(0)} \left( \hat{D}_{e_i^{\flat_\alpha}}^{(k)} \hat{Y} \right)(0) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \langle X_{f,\alpha} | e_{i_1} \otimes \cdots \otimes e_{i_k} \rangle_\alpha^\bullet \langle e_{i_1} \otimes \cdots \otimes e_{i_k} | Y \rangle_\alpha^\bullet \\ &= \langle X_{f,\alpha} | Y \rangle_\alpha^\bullet \end{aligned}$$

holds due to the construction of  $X_{f,\alpha}$  and Lemma 3.22 and because the tensors  $(k!)^{1/2} e_{i_1} \otimes \cdots \otimes e_{i_k}$  for all  $k \in \mathbb{N}_0$  and all  $i \in I^k$  are a Hilbert base of  $\mathcal{T}^\bullet(V)_\alpha^{\text{cpl}}$ .

Next, let  $\langle \cdot | \cdot \rangle_\beta \in \mathcal{I}_{V,h}$  with  $\langle \cdot | \cdot \rangle_\beta \leq \langle \cdot | \cdot \rangle_\alpha$  and a Hilbert basis  $d \in (V_{h,\beta}^{\text{cpl}})^J$  of  $V_{h,\beta}^{\text{cpl}}$  indexed by a set  $J$  be given. Using the explicit formulas and the identity

$$\left( \hat{D}_{d_j^{\flat_\beta}}^{(k)} f \right)(0) = \frac{1}{k!} \sum_{i \in I^k} \left( \hat{D}_{e_i^{\flat_\alpha}}^{(k)} f \right)(0) \langle d_{j_1} \otimes \cdots \otimes d_{j_k} | \iota_{\alpha\beta}(e_{i_1} \otimes \cdots \otimes e_{i_k}) \rangle_\beta^\bullet$$

from Lemma 3.18 one can now calculate that

$$\begin{aligned} \iota_{\alpha\beta}(X_{f,\alpha}) &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \iota_{\alpha\beta}(e_{i_1} \otimes \cdots \otimes e_{i_k}) \left( \hat{D}_{e_i^{\flat_\alpha}}^{(k)} f \right)(0) \\ &= \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \sum_{i \in I^k} \sum_{j \in J^k} d_{j_1} \otimes \cdots \otimes d_{j_k} \langle d_{j_1} \otimes \cdots \otimes d_{j_k} | \iota_{\alpha\beta}(e_{i_1} \otimes \cdots \otimes e_{i_k}) \rangle_\beta^\bullet \left( \hat{D}_{e_i^{\flat_\alpha}}^{(k)} f \right)(0) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j \in J^k} d_{j_1} \otimes \cdots \otimes d_{j_k} \left( \hat{D}_{d_j^{\flat_\beta}}^{(k)} f \right)(0) \\ &= X_{f,\beta}. \end{aligned}$$

As  $\mathcal{S}^\bullet(V)^{\text{cpl}}$  is the projective limit of the Hilbert spaces  $\mathcal{S}^\bullet(V)_\alpha^{\text{cpl}}$ , this implies that there exists a unique  $X_f \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  that fulfills  $\iota_{\infty\alpha}(X_f) = X_{f,\alpha}$  for all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$ . Consequently and with the help of Proposition 3.23,

$$\langle\langle \hat{X}_f | \hat{Y} \rangle\rangle_\alpha^\bullet(0) = \langle X_f | Y \rangle_\alpha^\bullet = \langle \iota_{\infty\alpha}(X_f) | Y \rangle_\alpha^\bullet = \langle X_{f,\alpha} | Y \rangle_\alpha^\bullet = \langle\langle f | \hat{Y} \rangle\rangle_\alpha^\bullet(0)$$

holds for all  $Y \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  and all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$ , and similarly,

$$\langle\langle \hat{X}_f | \hat{X}_f \rangle\rangle_\alpha^\bullet(0) = \langle X_f | X_f \rangle_\alpha^\bullet = \langle \iota_{\infty\alpha}(X_f) | \iota_{\infty\alpha}(X_f) \rangle_\alpha^\bullet = \langle X_{f,\alpha} | X_{f,\alpha} \rangle_\alpha^\bullet = \langle\langle f | f \rangle\rangle_\alpha^\bullet(0). \quad \square$$

After this preparation we are now able to identify the image of the Gel'fand transform explicitly:

**Theorem 3.26** *Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$ , then the Gel'fand transformation  $\hat{\cdot}: (\mathcal{S}^\bullet(V)^{\text{cpl}}, \vee, *) \rightarrow \mathcal{C}^{\omega HS}(V'_h)$  is an isomorphism of unital  $*$ -algebras.*

PROOF: Let  $X \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  be given, then  $\hat{X} \in \mathcal{C}^{\omega HS}(V'_h)$  by Corollary 3.24. The Gel'fand transformation is a unital  $*$ -homomorphism onto its image by construction and injective because  $\hat{X} = 0$  implies  $\langle X | X \rangle_\alpha^\bullet = \langle\langle \hat{X} | \hat{X} \rangle\rangle_\alpha^\bullet(0) = 0$  for all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  by Proposition 3.23, hence  $X = 0$ . It only remains to show that  $\hat{\cdot}$  is surjective, so let  $f \in \mathcal{C}^{\omega HS}(V'_h)$  be given. Construct  $X_f \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  like in the previous Lemma 3.25, then

$$\langle\langle f - \hat{X}_f | f - \hat{X}_f \rangle\rangle_\alpha^\bullet(0) = \langle\langle f | f \rangle\rangle_\alpha^\bullet(0) - \langle\langle f | \hat{X}_f \rangle\rangle_\alpha^\bullet(0) - \langle\langle \hat{X}_f | f \rangle\rangle_\alpha^\bullet(0) + \langle\langle \hat{X}_f | \hat{X}_f \rangle\rangle_\alpha^\bullet(0)$$

$$\begin{aligned}
&= \langle\langle f | f \rangle\rangle_\alpha^\bullet(0) - \langle\langle \widehat{X}_f | \widehat{X}_f \rangle\rangle_\alpha^\bullet(0) - \langle\langle \widehat{X}_f | \widehat{X}_f \rangle\rangle_\alpha^\bullet(0) + \langle\langle \widehat{X}_f | \widehat{X}_f \rangle\rangle_\alpha^\bullet(0) \\
&= 0
\end{aligned}$$

holds for all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$ , hence  $f = \widehat{X}_f$  due to Proposition 3.20.  $\square$

**Remark 3.27** Let  $\overline{\cdot}$  be a continuous antilinear involution on  $V$ . For a continuous bilinear form  $\Lambda$  on  $V$  the identity

$$P_\Lambda(X \otimes_\pi Y) = \sum_{i,i' \in I} \Lambda(e_i, e_{i'}) (D_{e_i} X \otimes_\pi D_{e_{i'}} Y) \quad (3.29)$$

holds for all  $X, Y \in \mathcal{S}^\bullet(V)$  and every  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  for which  $\|\cdot\|_\alpha \in \mathcal{P}_{V,\Lambda}$  and for every Hilbert base  $e \in (V_{h,\alpha}^{\text{cpl}})^I$  indexed by a set  $I$ . Thus

$$\widehat{X} \widehat{\star}_\Lambda \widehat{Y} := \widehat{X \star_\Lambda Y} = \mu \left( \sum_{t=0}^{\infty} \frac{1}{t!} \left( \sum_{i,i' \in I} \Lambda(e_i, e_{i'}) (\widehat{D}_{e_i} \otimes \widehat{D}_{e_{i'}}) \right)^t (\widehat{X} \otimes \widehat{Y}) \right)$$

with  $\mu: \mathcal{C}^\infty(V'_h) \otimes \mathcal{C}^\infty(V'_h) \rightarrow \mathcal{C}^\infty(V'_h)$  the pointwise product is the usual exponential star product on  $\mathcal{C}^{\omega_{HS}}(V'_h)$ . Moreover, if  $\mathcal{A} \subseteq \mathcal{C}^\infty(V'_h)$  is any unital  $\ast$ -subalgebra on which all such products  $\widehat{\star}_\Lambda$  for all continuous Hermitian bilinear forms  $\Lambda$  on  $V$  converge, then  $\mathcal{A} \subseteq \mathcal{C}^{\omega_{HS}}(V'_h)$ , because analogous to Proposition 3.23, every  $f \in \mathcal{A}$  fulfills  $\langle\langle f | f \rangle\rangle_\alpha^\bullet = f \widehat{\star}_{\Lambda_\alpha} f \in \mathcal{A} \subseteq \mathcal{C}^\infty(V'_h)$  for all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  with corresponding continuous Hermitian bilinear form  $V^2 \ni (v, w) \mapsto \Lambda_\alpha(v, w) := \langle \overline{v} | w \rangle_\alpha \in \mathbb{C}$ . This is of course just our Theorem 3.5 again.

### 3.4 Existence of continuous positive linear functionals

Recall that a linear functional  $\omega: \mathcal{S}^\bullet(V) \rightarrow \mathbb{C}$  is said to be positive for  $\star_\Lambda$  if  $\omega(X^\ast \star_\Lambda X) \geq 0$  holds for all  $X \in \mathcal{S}^\bullet(V)$ . Such positive linear functionals yield important information about the representation theory of a  $\ast$ -algebra, e.g. there exists a faithful  $\ast$ -representation as adjointable operators on a pre-Hilbert space if and only if the positive linear functionals are point-separating, see [20, Chap. 8.6]. In this section we will determine the obstructions for the existence of continuous positive linear functionals. First, we need the following lemma which allows us to apply an argument similar to the one used in [5] in the formal case:

**Lemma 3.28** *Let  $\overline{\cdot}$  be a continuous antilinear involution of  $V$  and  $\Lambda$  a continuous Hermitian bilinear form on  $V$  such that  $\Lambda(\overline{v}, v) \geq 0$  holds for all  $v \in V$ . Then for all  $X \in \mathcal{S}^\bullet(V)$  and all  $t \in \mathbb{N}_0$  there exist  $n \in \mathbb{N}$  and  $X_1, \dots, X_n \in \mathcal{S}^\bullet(V)$  such that*

$$(P_\Lambda)^t(X^\ast \otimes_\pi X) = \sum_{i=1}^n X_i^\ast \otimes_\pi X_i. \quad (3.30)$$

PROOF: This is trivial for scalar  $X$  as well as for  $t = 0$  and for the remaining cases it is sufficient to consider  $t = 1$ , the others then follow by induction. So let  $k \in \mathbb{N}$  and  $X \in \mathcal{S}^k(V)$  be given. Expand  $X$  as  $X = \sum_{j=1}^m x_{j,1} \vee \dots \vee x_{j,k}$  with  $m \in \mathbb{N}$  and vectors  $x_{1,1}, \dots, x_{m,k} \in V$ . Then

$$P_\Lambda(X^\ast \otimes_\pi X) = \sum_{j', j=1}^m \sum_{\ell', \ell=1}^k \Lambda(\overline{x_{j', \ell'}}, x_{j, \ell}) (x_{j', 1} \vee \dots \widehat{x_{j', \ell'}} \dots \vee x_{j', k})^\ast \otimes_\pi (x_{j, 1} \vee \dots \widehat{x_{j, \ell}} \dots \vee x_{j, k}),$$

where  $\widehat{\cdot}$  denotes omission of a vector in the product. The complex  $mk \times mk$ -matrix with entries  $\Lambda(\overline{x_{j', \ell'}}, x_{j, \ell})$  is positive due to the positivity condition on  $\Lambda$ , which implies that it has a Hermitian

square root  $R \in \mathbb{C}^{mk \times mk}$  that fulfills  $\Lambda(\overline{x_{j',\ell'}}, x_{j,\ell}) = \sum_{p=1}^m \sum_{q=1}^k \overline{R_{(p,q),(j',\ell')}} R_{(p,q),(j,\ell)}$  for all  $j, j' \in \{1, \dots, m\}$  and  $\ell, \ell' \in \{1, \dots, k\}$ . Consequently,

$$\begin{aligned} P_\Lambda(X^* \otimes_\pi X) &= \\ &= \sum_{p,q=1}^{m,k} \left( \sum_{j',\ell'=1}^{m,k} \overline{R_{(p,q),(j',\ell')}} (x_{j',1} \vee \cdots \widehat{x_{j',\ell'}} \cdots \vee x_{j',k})^* \right) \otimes_\pi \left( \sum_{j,\ell=1}^{m,k} R_{(p,q),(j,\ell)} (x_{j,1} \vee \cdots \widehat{x_{j,\ell}} \cdots \vee x_{j,k}) \right) \end{aligned}$$

holds which proves the lemma.  $\square$

**Proposition 3.29** *Let  $\overline{\cdot}$  be a continuous antilinear involution of  $V$  and  $\Lambda, \Lambda'$  as well as  $b$  three continuous Hermitian bilinear forms on  $V$  such that  $b$  is symmetric and of Hilbert-Schmidt type and such that  $\Lambda'(\overline{v}, v) + b(\overline{v}, v) \geq 0$  holds for all  $v \in V$ . Given a continuous linear functional  $\omega$  on  $\mathcal{S}^\bullet(V)$  that is positive for  $\star_\Lambda$ , define  $\omega_{zb}: \mathcal{S}^\bullet(V) \rightarrow \mathbb{C}$  as*

$$X \mapsto \omega_{zb}(X) := \omega(e^{z\Delta_b} X) \quad (3.31)$$

for all  $z \in \mathbb{R}$ . Then  $\omega_{zb}$  is a continuous linear functional and positive for  $\star_{\Lambda+z\Lambda'}$ .

PROOF: It follows from Theorem 3.10 that  $\omega_{zb}$  is continuous, and given  $X \in \mathcal{S}^\bullet(V)$ , then

$$\begin{aligned} \omega(e^{z\Delta_b}(X^* \star_{\Lambda+z\Lambda'} X)) &= \omega((e^{z\Delta_b} X)^* \star_{\Lambda+z(\Lambda'+b)} (e^{z\Delta_b} X)) \\ &= \sum_{s,t=0}^{\infty} \frac{1}{s!t!} \omega\left(\mu_\vee\left((P_\Lambda)^s (P_{z(\Lambda'+b)})^t ((e^{z\Delta_b} X)^* \otimes_\pi (e^{z\Delta_b} X))\right)\right) \\ &= \sum_{t=0}^{\infty} \frac{1}{t!} \omega\left(\mu_{\star_\Lambda}\left((P_{z(\Lambda'+b)})^t ((e^{z\Delta_b} X)^* \otimes_\pi (e^{z\Delta_b} X))\right)\right) \\ &\geq 0 \end{aligned}$$

holds because  $P_\Lambda$  and  $P_{z(\Lambda'+b')}$  commute on symmetric tensors and because of Lemma 3.28.  $\square$

Note that Theorem 3.10 also shows that  $\omega_{zb}$  depends holomorphically on  $z \in \mathbb{C}$  in so far as  $\mathbb{C} \ni z \mapsto \omega_{zb}(X) \in \mathbb{C}$  is holomorphic for all  $X \in \mathcal{S}^\bullet(V)$ . This is the analog of statements in [12, 13] in the Rieffel setting.

**Proposition 3.30** *Let  $\overline{\cdot}$  be a continuous antilinear involution of  $V$  and  $\Lambda$  a continuous Hermitian bilinear forms on  $V$ . If there exists a continuous linear functional  $\omega$  on  $\mathcal{S}^\bullet(V)$  that is positive for  $\star_\Lambda$  and fulfills  $\omega(1) = 1$ , then the bilinear form  $V^2 \ni (v, w) \mapsto b_\omega(v, w) := \omega(v \vee w) \in \mathbb{C}$  is symmetric, Hermitian, of Hilbert-Schmidt type and fulfills  $\Lambda(\overline{v}, v) + b_\omega(\overline{v}, v) \geq 0$  for all  $v \in V$ .*

PROOF: It follows immediately from the construction of  $b_\omega$  that this bilinear form is symmetric and it is Hermitian because  $\overline{b_\omega(v, w)} = \omega(\overline{v \vee w}) = \omega(\overline{v} \vee \overline{w}) = b_\omega(\overline{w}, \overline{v})$  holds for all  $v, w \in V$ . Continuity of  $\omega$  especially implies that there exists a  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_V$  such that  $|\omega(X)| \leq 2^{-1/2} \|X\|_\alpha^\bullet$  holds for all  $X \in \mathcal{S}^2(V)$ , hence  $b_\omega$  is of Hilbert-Schmidt type by Proposition 3.8 and because  $\Delta_{b_\omega} X = \omega(X)$  for  $X \in \mathcal{S}^2(V)$ . Finally,  $0 \leq \omega(v^* \star_\Lambda v) = \Lambda(\overline{v}, v) + b_\omega(\overline{v}, v)$  holds due to the positivity of  $\omega$ .  $\square$

**Theorem 3.31** *Let  $\overline{\cdot}$  be a continuous antilinear involution of  $V$  and  $\Lambda$  a continuous Hermitian bilinear forms on  $V$ . Assume  $V \neq \{0\}$ . There exists a non-zero continuous positive linear functional on  $(\mathcal{S}^\bullet(V), \star_\Lambda, *)$  if and only if there exists a symmetric and hermitian bilinear form of Hilbert-Schmidt type  $b$  on  $V$  such that  $\Lambda(\overline{v}, v) + b(\overline{v}, v) \geq 0$  holds for all  $v \in V$ . In this case, the continuous positive linear functionals on  $(\mathcal{S}^\bullet(V), \star_\Lambda, *)$  are point-separating, i.e. their common kernel is  $\{0\}$ .*



PROOF: If there exists a non-zero continuous positive linear functional  $\omega$  on  $(\mathcal{S}^\bullet(V), \star_\Lambda, *)$ , then  $\omega(1) \neq 0$  due to the Cauchy-Schwarz identity and we can rescale  $\omega$  such that  $\omega(1) = 1$ . Then the previous Proposition 3.30 shows the existence of such a bilinear form  $b$ . Conversely, if such a bilinear form  $b$  exists, then Proposition 3.29 shows that all continuous linear functionals on  $\mathcal{S}^\bullet(V)$  that are positive for  $\vee$  can be deformed to continuous linear functionals that are positive for  $\star_\Lambda$  by taking the pull-back with  $e^{\Delta_b}$ . As  $e^{\Delta_b}$  is invertible, it only remains to show that the continuous positive linear functionals on  $(\mathcal{S}^\bullet(V), \vee, *)$  are point-separating. This is an immediate consequence of Theorem 3.26, which especially shows that the evaluation functionals  $\delta_\rho$  with  $\rho \in V'_h$  are point-separating.  $\square$

### 3.5 Exponentials and Essential Self-Adjointness in GNS Representations

Having a topology on the symmetric tensor algebra allows us to ask whether or not some exponentials (with respect to the undeformed or deformed products) exist in the completion, i.e. we want to discuss for which tensors  $X \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  the series  $\exp_{\star_\Lambda}(X) := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\star_\Lambda n}$  converges, where  $X^{\star_\Lambda n}$  denotes the  $n$ -th power of  $X$  with respect to the product  $\star_\Lambda$  for a continuous bilinear form  $\Lambda$  on  $V$ . Note that since the algebra is (necessarily) *not* locally multiplicatively convex, this is a non-trivial question. This also allows to give a sufficient criterium for a GNS representation of a Hermitian algebra element to be essentially self-adjoint.

**Definition 3.32** For  $k \in \mathbb{N}_0$  we define

$$\mathcal{S}^{(k)}(V) := \bigoplus_{\ell=0}^k \mathcal{S}^\ell(V), \quad (3.32)$$

and write  $\mathcal{S}^{(k)}(V)^{\text{cpl}}$  for the closure of  $\mathcal{S}^{(k)}(V)$  in  $\mathcal{S}^\bullet(V)^{\text{cpl}}$ .

**Lemma 3.33** One has

$$\binom{m}{\ell} \binom{m-\ell+t}{t} \leq \binom{\ell+t}{t} \binom{k(n+1)}{k} \quad (3.33)$$

for all  $k, n \in \mathbb{N}_0$ ,  $m \in \{0, \dots, kn\}$ ,  $t \in \{0, \dots, k\}$ , and all  $\ell \in \{0, \dots, \min\{m, k-t\}\}$ .

**Lemma 3.34** Let  $\Lambda$  be a continuous bilinear form on  $V$ . Let  $k, n \in \mathbb{N}_0$  and  $X_1, \dots, X_n \in \mathcal{S}^{(k)}(V)^{\text{cpl}}$  be given. Then the estimates

$$\|\langle X_1 \star_\Lambda \dots \star_\Lambda X_n \rangle_m\|_\alpha^\bullet \leq \left( \frac{(kn)!}{(k!)^n} \right)^{\frac{1}{2}} (2e^2)^{kn} \|X_1\|_\alpha^\bullet \dots \|X_n\|_\alpha^\bullet \quad (3.34)$$

and

$$\|X_1 \star_\Lambda \dots \star_\Lambda X_n\|_\alpha^\bullet \leq \left( \frac{(kn)!}{(k!)^n} \right)^{\frac{1}{2}} (2e^3)^{kn} \|X_1\|_\alpha^\bullet \dots \|X_n\|_\alpha^\bullet \quad (3.35)$$

hold for all  $m \in \{0, \dots, kn\}$  and all  $\|\cdot\|_\alpha \in \mathcal{P}_{V, \Lambda}$ .

PROOF: The first estimate implies the second, because  $\|X_1 \star_\Lambda \dots \star_\Lambda X_n\|_\alpha^\bullet$  has at most  $(1+kn)$  non-vanishing homogeneous components, namely those of degree  $m \in \{0, \dots, kn\}$ , and  $(1+kn) \leq e^{kn}$ . We will prove the first estimate by induction over  $n$ : If  $n = 0$  or  $n = 1$ , then the estimate is clearly fulfilled for all possible  $k$  and  $m$ , and if it holds for one  $n \in \mathbb{N}$ , then

$$\begin{aligned} & \|\langle X_1 \star_\Lambda \dots \star_\Lambda X_{n+1} \rangle_m\|_\alpha^\bullet \\ & \leq \sum_{t=0}^k \frac{1}{t!} \left\| \left\langle \mu_\vee \left( (P_\Lambda)^t ((X_1 \star_\Lambda \dots \star_\Lambda X_n) \otimes_\pi X_{n+1}) \right) \right\rangle_m \right\|_\alpha^\bullet \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{t=0}^k \sum_{\ell=0}^{\min\{m, k-t\}} \frac{1}{t!} \left\| \mu_{\vee} \left( (P_{\Lambda})^t (\langle X_1 \star_{\Lambda} \cdots \star_{\Lambda} X_n \rangle_{m-\ell+t} \otimes_{\pi} \langle X_{n+1} \rangle_{\ell+t}) \right) \right\|_{\alpha}^{\bullet} \\
&\leq \sum_{t=0}^k \sum_{\ell=0}^{\min\{m, k-t\}} \frac{1}{t!} \binom{m}{\ell}^{\frac{1}{2}} \left\| (P_{\Lambda})^t (\langle X_1 \star_{\Lambda} \cdots \star_{\Lambda} X_n \rangle_{m-\ell+t} \otimes_{\pi} \langle X_{n+1} \rangle_{\ell+t}) \right\|_{\alpha \otimes_{\pi} \alpha}^{\bullet} \\
&\leq \sum_{t=0}^k \sum_{\ell=0}^{\min\{m, k-t\}} \binom{m}{\ell}^{\frac{1}{2}} \binom{m-\ell+t}{t}^{\frac{1}{2}} \binom{\ell+t}{t}^{\frac{1}{2}} \|\langle X_1 \star_{\Lambda} \cdots \star_{\Lambda} X_n \rangle_{m-\ell+t}\|_{\alpha}^{\bullet} \|\langle X_{n+1} \rangle_{\ell+t}\|_{\alpha}^{\bullet} \\
&\leq \sum_{t=0}^k \sum_{\ell=0}^{\min\{m, k-t\}} \binom{\ell+t}{t} \binom{k(n+1)}{k}^{\frac{1}{2}} \|\langle X_1 \star_{\Lambda} \cdots \star_{\Lambda} X_n \rangle_{m-\ell+t}\|_{\alpha}^{\bullet} \|\langle X_{n+1} \rangle_{\ell+t}\|_{\alpha}^{\bullet} \\
&\leq \sum_{t=0}^k \sum_{\ell=0}^{\min\{m, k-t\}} \binom{\ell+t}{t} \binom{k(n+1)}{k}^{\frac{1}{2}} \left( \frac{(kn)!}{(k!)^n} \right)^{\frac{1}{2}} (2e^2)^{kn} \|X_1\|_{\alpha}^{\bullet} \cdots \|X_n\|_{\alpha}^{\bullet} \|X_{n+1}\|_{\alpha}^{\bullet} \\
&= \sum_{t=0}^k \sum_{\ell=0}^{\min\{m, k-t\}} \binom{\ell+t}{t} \left( \frac{(k(n+1))!}{(k!)^{n+1}} \right)^{\frac{1}{2}} (2e^2)^{kn} \|X_1\|_{\alpha}^{\bullet} \cdots \|X_{n+1}\|_{\alpha}^{\bullet} \\
&\leq \left( \frac{(k(n+1))!}{(k!)^{n+1}} \right)^{\frac{1}{2}} (2e^2)^{k(n+1)} \|X_1\|_{\alpha}^{\bullet} \cdots \|X_{n+1}\|_{\alpha}^{\bullet}
\end{aligned}$$

holds due to the grading of  $\mu_{\vee}$  and  $P_{\Lambda}$ , the estimates from Propositions 2.6 as well as 2.7 and Lemma 2.10 for  $\mu_{\vee}$  and  $P_{\Lambda}$ , and the previous Lemma 3.33.  $\square$

**Proposition 3.35** *Let  $\Lambda$  be a continuous bilinear form on  $V$ , then  $\exp_{\star_{\Lambda}}(v)$  is absolutely convergent and*

$$\exp_{\star_{\Lambda}}(v) = \sum_{n=0}^{\infty} \frac{v^{\star_{\Lambda} n}}{n!} = e^{\frac{1}{2}\Lambda(v,v)} \exp_{\vee}(v) \quad (3.36)$$

holds for all  $v \in V$ . Moreover,

$$\exp_{\vee}(v) \star_{\Lambda} \exp_{\vee}(w) = e^{\Lambda(v,w)} \exp_{\vee}(v+w) \quad (3.37)$$

and

$$\langle \exp_{\vee}(v) \mid \exp_{\vee}(w) \rangle_{\alpha}^{\bullet} = e^{\langle v \mid w \rangle_{\alpha}} \quad (3.38)$$

hold for all  $v, w \in V$  and all  $\langle \cdot \mid \cdot \rangle_{\alpha} \in \mathcal{I}_V$ . Finally,  $\exp_{\vee}(v)^* = \exp_{\vee}(\bar{v})$  for all  $v \in V$  if  $V$  is equipped with a continuous antilinear involution  $\bar{\cdot}$ .

PROOF: The existence and absolute convergence of  $\star_{\Lambda}$ -exponentials of vectors follows directly from the previous Lemma 3.34 with  $k = 1$  and  $X_1 = \cdots = X_n = v$ :

$$\sum_{n=0}^{\infty} \frac{\|v^{\star_{\Lambda} n}\|_{\alpha}^{\bullet}}{n!} \leq \sum_{n=0}^{\infty} \frac{(4e^3\|v\|_{\alpha})^n}{\sqrt{n!}} \frac{1}{2^n} \stackrel{\text{cs}}{\leq} \left( \sum_{n=0}^{\infty} \frac{(4e^3\|v\|_{\alpha})^{2n}}{n!} \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} \frac{1}{4^n} \right)^{\frac{1}{2}} = \frac{2e^{16e^6\|v\|_{\alpha}^2}}{\sqrt{3}}$$

The explicit formula can then be derived like in [23, Lem. 5.5]. For (3.37) we just note that

$$\begin{aligned}
P_{\Lambda}(\exp_{\vee}(v) \otimes_{\pi} \exp_{\vee}(w)) &= \sum_{k,\ell=0}^{\infty} P_{\Lambda} \left( \frac{v^{\vee k}}{k!} \otimes_{\pi} \frac{w^{\vee \ell}}{\ell!} \right) \\
&= \Lambda(v, w) \sum_{k,\ell=1}^{\infty} \frac{kv^{\vee(k-1)}}{k!} \otimes_{\pi} \frac{\ell w^{\vee(\ell-1)}}{\ell!}
\end{aligned}$$

$$= \Lambda(v, w) \exp_V(v) \otimes_\pi \exp_V(w),$$

and so

$$\exp_V(v) \star_\Lambda \exp_V(w) = \sum_{t=0}^{\infty} \frac{1}{t!} \mu_V \left( (P_\Lambda)^t (\exp_V(v) \otimes_\pi \exp_V(w)) \right) = e^{\Lambda(v, w)} \exp_V(v) \vee \exp_V(w).$$

The remaining two identities are the results of straightforward calculations.  $\square$

As an application we show that there exists a dense  $\ast$ -subalgebra consisting of uniformly bounded elements:

**Definition 3.36** *Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$ . We define the linear subspace*

$$\mathcal{S}_{\text{per}}^\bullet(V) := \text{span} \{ \exp_V(iv) \in \mathcal{S}^\bullet(V)^{\text{cpl}} \mid v \in V \text{ and } \bar{v} = v \} \quad (3.39)$$

*of  $\mathcal{S}^\bullet(V)^{\text{cpl}}$ .*

**Proposition 3.37** *Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$ . Then  $\mathcal{S}_{\text{per}}^\bullet(V)$  is a dense  $\ast$ -subalgebra of  $(\mathcal{S}^\bullet(V)^{\text{cpl}}, \star_\Lambda, \ast)$  with respect to all products  $\star_\Lambda$  for all continuous bilinear Hermitian forms  $\Lambda$  on  $V$  and*

$$\|X\|_{\infty, \Lambda} := \sup \sqrt{\omega(X^\ast \star_\Lambda X)} < \infty \quad (3.40)$$

*holds for all  $X \in \mathcal{S}_{\text{per}}^\bullet(V)$ , where the supremum runs over all continuous positive linear functionals  $\omega$  on  $(\mathcal{S}^\bullet(V), \star_\Lambda, \ast)$  that are normalized to  $\omega(1) = 1$ .*

PROOF: Proposition 3.35 shows that  $\mathcal{S}_{\text{per}}^\bullet(V)$  is a  $\ast$ -subalgebra of  $\mathcal{S}^\bullet(V)^{\text{cpl}}$  with respect to all products  $\star_\Lambda$  for all continuous bilinear Hermitian forms  $\Lambda$  on  $V$ . As  $-i \frac{d}{dz} \big|_{z=0} \exp_V(izv) = v$  for all  $v \in V$  with  $v = \bar{v}$  we see that the closure of the subalgebra  $\mathcal{S}_{\text{per}}^\bullet(V)$  contains  $\bar{V}$ , hence  $\mathcal{S}^\bullet(V)$  which is (as a unital algebra) generated by  $V$ , and so the closure of  $\mathcal{S}_{\text{per}}^\bullet(V)$  coincides with  $\mathcal{S}^\bullet(V)^{\text{cpl}}$ .

As  $\mathcal{S}_{\text{per}}^\bullet(V)$  is spanned by exponentials and  $\omega(\exp_V(iv)^\ast \star_\Lambda \exp_V(iv)) = e^{\Lambda(v, v)} \omega(\exp_V(0)) = e^{\Lambda(v, v)}$  holds for all positive linear functionals  $\omega$  on  $(\mathcal{S}^\bullet(V), \star_\Lambda, \ast)$  that are normalized to  $\omega(1) = 1$  by Proposition 3.35, it follows that  $\|X\|_{\infty, \Lambda} < \infty$  for all  $X \in \mathcal{S}_{\text{per}}^\bullet(V)$ .  $\square$

Note that one can show that  $\|\cdot\|_{\infty, \Lambda}$  is a  $C^\ast$ -norm on  $(\mathcal{S}^\bullet(V), \star_\Lambda, \ast)$  if the continuous positive linear functionals are point-separating. In contrast to the existence of exponential of vectors, we get strict constraints on the existence of exponentials of quadratic elements:

**Proposition 3.38** *Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$ . Then there is no locally convex topology  $\tau$  on  $\mathcal{S}_{\text{alg}}^\bullet(V)$  with the property that any (undeformed) exponential  $\exp_V(X) = \sum_{n=0}^{\infty} \frac{X^{\vee n}}{n!}$  of any  $X \in \mathcal{S}^2(V) \setminus \{0\}$  exists in the completion of  $\mathcal{S}_{\text{alg}}^\bullet(V)$  under  $\tau$  and such that all the products  $\star_\Lambda$  for all continuous Hermitian bilinear forms  $\Lambda$  on  $V$  as well as the  $\ast$ -involution and the projection  $\langle \cdot \rangle_0$  on the scalars are continuous.*

PROOF: Analogously to the proof of Theorem 3.5 we see that, if all the products  $\star_\Lambda$  for all continuous Hermitian bilinear forms  $\Lambda$  on  $V$  as well as the  $\ast$ -involution and the projection  $\langle \cdot \rangle_0$  on the scalars are continuous, then all the extended positive Hermitian forms  $\langle \cdot | \cdot \rangle_\alpha^\bullet$  for all  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_V$  would have to be continuous and thus extend to the completion of  $\mathcal{S}_{\text{alg}}^\bullet(V)$ .

Now let  $X \in \mathcal{S}^2(V) \setminus \{0\}$  be given. There exist  $k \in \mathbb{N}$  and  $x \in V^k$  such that  $x_1, \dots, x_k$  are linearly independent and  $X = \sum_{i=1}^k \sum_{j=i}^k \tilde{X}^{ij} x_i \vee x_j$  with complex coefficients  $\tilde{X}^{ij}$ . If there exists an  $i \in \{1, \dots, k\}$  such that  $\tilde{X}^{ii} \neq 0$ , then we can assume without loss of generality that  $i = 1$  and  $\tilde{X}^{11} = 1$  and define a continuous positive Hermitian form on  $V$  by  $\langle v | w \rangle_\omega := \overline{\omega(v)} \omega(w)$ , where  $\omega: V \rightarrow \mathbb{C}$  is

a continuous linear form on  $V$  that satisfies  $\omega(x_1) = 1$  and  $\omega(x_i) = 0$  for  $i \in \{2, \dots, k\}$ . Otherwise we can assume without loss of generality that  $\tilde{X}^{11} = \tilde{X}^{22} = 0$  and  $\tilde{X}^{12} = 1$  and define a continuous positive Hermitian form on  $V$  by  $\langle v | w \rangle_\omega := \overline{\omega(v)}^T \omega(w)$ , where  $\omega: V \rightarrow \mathbb{C}^2$  is a continuous linear map that satisfies  $\omega(x_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\omega(x_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $\omega(x_i) = 0$  for  $i \in \{3, \dots, k\}$ .

In the first case, this results in  $\langle X^{\vee n} | X^{\vee n} \rangle_\omega^\bullet = (2n)!$  and in the second,  $\langle X^{\vee n} | X^{\vee n} \rangle_\omega^\bullet = (n!)^2$ . So  $\sum_{n=0}^\infty \frac{X^{\vee n}}{n!}$  cannot converge in the completion of  $\mathcal{S}_{\text{alg}}^\bullet(V)$  because

$$\left\langle \sum_{n=0}^N \frac{X^{\vee n}}{n!} \middle| \sum_{n=0}^N \frac{X^{\vee n}}{n!} \right\rangle_\omega^\bullet \geq \sum_{n=0}^N 1 \xrightarrow{N \rightarrow \infty} \infty. \quad \square$$

A similar result has already been obtained by Omori, Maeda, Miyazaki and Yoshioka in the 2-dimensional case in [16], where they show that associativity of the Moyal-product breaks down on exponentials of quadratic functions. Note that the above proposition does not exclude the possibility that exponentials of *some* quadratic functions exist if one only demands that *some* special deformations are continuous.

Even though exponentials of non-trivial tensors of degree 2 are not contained in  $\mathcal{S}^\bullet(V)^{\text{cpl}}$ , the continuous positive linear functionals are in some sense “analytic” for such tensors:

**Proposition 3.39** *Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$  and  $\Lambda$  a continuous Hermitian bilinear form on  $V$ . Let  $\omega: \mathcal{S}^\bullet(V)^{\text{cpl}} \rightarrow \mathbb{C}$  be a continuous linear functional on  $\mathcal{S}^\bullet(V)^{\text{cpl}}$  that is positive with respect to  $\star_\Lambda$ . Then for all  $X \in \mathcal{S}^{(2)}(V)^{\text{cpl}}$  there exists an  $\epsilon > 0$  such that*

$$\sum_{n=0}^\infty \frac{\epsilon^n \omega((X \star_\Lambda n)^* \star_\Lambda X \star_\Lambda n)^{\frac{1}{2}}}{n!} < \infty \quad (3.41)$$

holds.

PROOF: The seminorm  $\mathcal{S}^\bullet(V)^{\text{cpl}} \ni Y \mapsto \omega(Y^* \star_\Lambda Y)^{1/2} \in [0, \infty[$  is continuous by construction, so there exist  $C > 0$  and  $\|\cdot\|_\alpha \in \mathcal{P}_V$  such that  $\omega(Y^* \star_\Lambda Y)^{1/2} \leq C \|Y\|_\alpha^\bullet$  holds for all  $Y \in \mathcal{S}^\bullet(V)^{\text{cpl}}$ . We can even assume without loss of generality that  $\|\cdot\|_\alpha \in \mathcal{P}_{V, \Lambda}$ . Now choose  $\epsilon > 0$  with  $\epsilon(8e^6 \|X\|_\alpha^{\bullet, 2}) \leq 1$ , then Lemma 3.34 in the case  $k = 2$  and  $X_1 = \dots = X_n = X$  shows that

$$\begin{aligned} \sum_{n=0}^\infty \frac{\epsilon^n \omega((X \star_\Lambda n)^* \star_\Lambda X \star_\Lambda n)^{\frac{1}{2}}}{n!} &\leq C \sum_{n=0}^\infty \frac{\epsilon^n \|X \star_\Lambda n\|_\alpha^\bullet}{n!} \\ &\leq C \sum_{n=0}^\infty \frac{\sqrt{(2n)!}}{\sqrt{2}^{3n} n!} \\ &\leq C \sum_{n=0}^\infty \frac{1}{\sqrt{2}^n} \\ &= \frac{C\sqrt{2}}{\sqrt{2}-1}. \end{aligned} \quad \square$$

It is an immediate consequence of this proposition that Hermitian tensors of grade at most 2 are represented by essentially self-adjoint operators in every GNS-representation corresponding to a continuous positive linear functional  $\omega$ . Recall that for a  $*$ -algebra  $\mathcal{A}$  with a positive linear functional  $\omega: \mathcal{A} \rightarrow \mathbb{C}$ , the GNS representation of  $\mathcal{A}$  associated to  $\omega$  is the unital  $*$ -homomorphism  $\pi_\omega: \mathcal{A} \rightarrow \text{Adj}(\mathcal{A}/\mathcal{I}_\omega)$  into the adjointable endomorphisms on the pre-Hilbert space  $\mathcal{H}_\omega = \mathcal{A}/\mathcal{I}_\omega$  with inner product  $\langle \cdot | \cdot \rangle_\omega$ , where  $\mathcal{I}_\omega = \{a \in \mathcal{A} \mid \omega(a^*a) = 0\}$  and  $\langle [a] | [b] \rangle_\omega = \omega(a^*b)$  for all  $[a], [b] \in \mathcal{H}_\omega$  with representatives  $a, b \in \mathcal{A}$ .

**Theorem 3.40** *Let  $\bar{\cdot}$  be a continuous antilinear involution on  $V$  and  $\Lambda$  a continuous Hermitian bilinear form on  $V$ . Let  $\omega: \mathcal{S}^\bullet(V)^{\text{cpl}} \rightarrow \mathbb{C}$  be a continuous linear functional on  $\mathcal{S}^\bullet(V)^{\text{cpl}}$  that is positive with respect to  $\star_\Lambda$ . Then for  $X^* = X \in \mathcal{S}^{(2)}(V)^{\text{cpl}}$  all vectors in the GNS pre-Hilbert space  $\mathcal{H}_\omega$  are analytic for  $\pi_\omega(X)$  which is therefore essentially self-adjoint.*

PROOF: It is clear from the construction of the GNS representation that  $\pi_\omega(X)$  is a symmetric operator on  $\mathcal{H}_\omega = \mathcal{S}^\bullet(V)^{\text{cpl}}/\mathcal{I}_\omega$  and by Nelson's theorem, see e.g. [21, Thm. 7.16], it is sufficient to show that all vectors  $[Y] \in \mathcal{H}_\omega$  are analytic for  $\pi_\omega(X)$ : From

$$\langle \pi_\omega(X)^n[Y] | \pi_\omega(X)^n[Y] \rangle_\omega = \omega((X \star_\Lambda^n Y)^* \star_\Lambda (X \star_\Lambda^n Y)) = \omega(Y^* \star_\Lambda (X \star_\Lambda^n)^* \star_\Lambda X \star_\Lambda^n Y)$$

it follows that analyticity of the vector  $[Y]$  is equivalent to the analyticity of the continuous positive linear functional  $\mathcal{S}^\bullet(V)^{\text{cpl}} \ni Z \mapsto \omega_Y(Z) := \omega(Y^* \star_\Lambda Z \star_\Lambda Y) \in \mathbb{C}$  in the sense of the previous Proposition 3.39.  $\square$

## 4 Special Cases and Examples

Finally we want to discuss two special cases that have appeared in the literature before, namely that  $V$  is a Hilbert space and that  $V$  is a nuclear space.

### 4.1 Deformation Quantization of Hilbert Spaces

Assume that  $V$  is a (complex) Hilbert space with inner product  $\langle \cdot | \cdot \rangle_1$ . We note that in this case  $\mathcal{S}^\bullet(V)$  is not a pre-Hilbert space but only a countable projective limit of pre-Hilbert spaces, because the extensions  $\langle \cdot | \cdot \rangle_\alpha^\bullet$  of the (equivalent) inner products  $\langle \cdot | \cdot \rangle_\alpha := \alpha \langle \cdot | \cdot \rangle_1$  for  $\alpha \in ]0, \infty[$  are not equivalent. If  $V$  is a Hilbert space, then its topological dual and, more generally, all spaces of bounded multilinear functionals on  $V$  are Banach spaces. This allows a more detailed analysis of the continuity of functions in  $\mathcal{C}^{\omega HS}(V'_h)$  and of the dependence of the product  $\star_\Lambda$  on  $\Lambda \in \mathfrak{Bil}(V)$ .

**Theorem 4.1** *Let  $V$  be a (complex) Hilbert space with inner product  $\langle \cdot | \cdot \rangle_1$  and unit ball  $U \subseteq V$  and let  $\mathfrak{Bil}(V)$  be the Banach space of all continuous bilinear forms on  $V$  with norm  $\|\Lambda\| := \sup_{v,w \in U} |\Lambda(v,w)|$ . Then the map  $\mathfrak{Bil}(V) \times \mathcal{S}^\bullet(V)^{\text{cpl}} \times \mathcal{S}^\bullet(V)^{\text{cpl}} \rightarrow \mathcal{S}^\bullet(V)^{\text{cpl}}$*

$$(\Lambda, X, Y) \mapsto X \star_\Lambda Y \tag{4.1}$$

*is continuous.*

PROOF: Note that for a Hilbert space  $V$ , the continuous inner products  $\langle \cdot | \cdot \rangle_\lambda$  with  $\lambda > 0$  are cofinal in  $\mathcal{I}_V$ . Now let  $\Lambda \in \mathfrak{Bil}(V)$ ,  $X, Y \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  and  $\epsilon > 0$  be given, then

$$\|X' \star_{\Lambda'} Y' - X \star_\Lambda Y\|_\lambda \leq \|X' \star_{\Lambda'} Y' - X \star_{\Lambda'} Y\|_\lambda + \|X \star_{\Lambda'} Y - X \star_\Lambda Y\|_\lambda$$

holds for all  $\lambda > 0$  and all  $\Lambda' \in \mathfrak{Bil}(V)$  as well as all  $X', Y' \in \mathcal{S}^\bullet(V)^{\text{cpl}}$ . Moreover,

$$\begin{aligned} \|X' \star_{\Lambda'} Y' - X \star_{\Lambda'} Y\|_\lambda &\leq \|(X' - X) \star_{\Lambda'} Y'\|_\lambda + \|X \star_{\Lambda'} (Y' - Y)\|_\lambda \\ &\leq 4\|X' - X\|_{8\lambda}^\bullet \|Y'\|_{8\lambda}^\bullet + 4\|X\|_{8\lambda}^\bullet \|Y' - Y\|_{8\lambda}^\bullet \end{aligned}$$

holds for all  $X', Y' \in \mathcal{S}^\bullet(V)^{\text{cpl}}$  as well as all  $\lambda > 0$  and all  $\Lambda' \in \mathfrak{Bil}(V)$  such that  $\|\cdot\|_\lambda \in \mathcal{P}_{V, \Lambda'}$  by Lemma 2.12. One can check on factorizing symmetric tensors that  $P_\Lambda$  and  $P_{\Lambda' - \Lambda}$  commute and by using that

$$X \star_{\Lambda'} Y = \sum_{t'=0}^{\infty} \frac{1}{t'!} \mu_V \left( (P_{\Lambda + (\Lambda' - \Lambda)})^{t'} (X \otimes_\pi Y) \right)$$

$$\begin{aligned}
&= \sum_{t,s=0}^{\infty} \frac{1}{t!s!} \mu_{\vee} \left( (P_{\Lambda})^t (P_{\Lambda' - \Lambda})^s (X \otimes_{\pi} Y) \right) \\
&= \sum_{s=0}^{\infty} \frac{1}{s!} \mu_{\star_{\Lambda}} \left( (P_{\Lambda' - \Lambda})^s (X \otimes_{\pi} Y) \right),
\end{aligned}$$

it follows that

$$\begin{aligned}
\|X \star_{\Lambda'} Y - X \star_{\Lambda} Y\|_{\lambda} &\leq \sum_{s=1}^{\infty} \frac{1}{\rho^s s!} \left\| \mu_{\star_{\Lambda}} \left( (P_{\rho(\Lambda' - \Lambda)})^s (X \otimes_{\pi} Y) \right) \right\|_{\lambda}^{\bullet} \\
&\leq 4 \sum_{s=1}^{\infty} \frac{1}{\rho^s s!} \left\| (P_{\rho(\Lambda' - \Lambda)})^s (X \otimes_{\pi} Y) \right\|_{8\lambda \otimes_{\pi} 8\lambda}^{\bullet} \\
&\leq 8 \sum_{s=1}^{\infty} \frac{1}{(2\rho)^s} \|X\|_{32\lambda}^{\bullet} \|Y\|_{32\lambda}^{\bullet} \\
&= \frac{8}{2\rho - 1} \|X\|_{32\lambda}^{\bullet} \|Y\|_{32\lambda}^{\bullet}
\end{aligned}$$

holds for all  $\rho > \frac{1}{2}$ ,  $\lambda > 0$ , and all  $\Lambda' \in \mathfrak{Bil}(V)$  if  $\|\cdot\|_{\lambda} \in \mathcal{P}_{V,\Lambda} \cap \mathcal{P}_{V,\rho(\Lambda' - \Lambda)}$  by Lemma 2.12 and Proposition 2.11 with  $c = 2$ .

Assume that  $\lambda \geq 1 + \|\Lambda\|$  and choose  $\rho > \frac{1}{2}$  such that  $\frac{8}{2\rho - 1} \|X\|_{32\lambda}^{\bullet} \|Y\|_{32\lambda}^{\bullet} \leq \frac{\epsilon}{3}$ . Then  $\|\cdot\|_{\lambda} \in \mathcal{P}_{V,\Lambda} \cap \mathcal{P}_{V,\rho(\Lambda' - \Lambda)}$  for all  $\Lambda' \in \mathfrak{Bil}(V)$  with  $\|\Lambda' - \Lambda\| \leq \frac{1}{\rho}$  and  $\|X' \star_{\Lambda'} Y' - X \star_{\Lambda} Y\|_{\lambda} \leq \epsilon$  holds for all these  $\Lambda'$  and all  $X', Y' \in \mathcal{S}^{\bullet}(V)^{\text{cpl}}$  with  $\|X' - X\|_{8\lambda}^{\bullet} \leq \epsilon/(12 + 12\|Y\|_{8\lambda}^{\bullet})$  and  $\|Y' - Y\|_{8\lambda} \leq \min\{1, \epsilon/(12 + 12\|X\|_{8\lambda}^{\bullet})\}$ . This proves continuity of  $\star$  at  $(\Lambda, X, Y)$ .  $\square$

**Theorem 4.2** *Let  $V$  be a (complex) Hilbert space with inner product  $\langle \cdot | \cdot \rangle_1$  and a continuous anti-linear involution  $\bar{\cdot}$  that fulfills  $\overline{\langle v | w \rangle_1} = \langle \bar{v} | \bar{w} \rangle_1$  for all  $v, w \in V$ , then  $\widehat{X}: V'_h \rightarrow \mathbb{C}$  is smooth in the Fréchet sense for all  $X \in \mathcal{S}^{\bullet}(V)^{\text{cpl}}$ .*

PROOF: By the Fréchet-Riesz theorem we can identify  $V'_h$  with  $V_h$  by means of the antilinear map  $\cdot^b: V_h \rightarrow V'_h$ . As the translations  $\tau^*$  are automorphisms of  $\mathcal{S}^{\bullet}(V)^{\text{cpl}}$ , it is sufficient to show that  $\widehat{X}$  is smooth at  $0 \in V'_h$ . So let  $K \in \mathbb{N}_0$  and  $r \in V_h$  be given with  $r \neq 0$  and  $\|r\|_1 \leq 1$ . We have already seen in Proposition 3.17 that all directional derivatives of  $\widehat{X}$  exist and form bounded symmetric multilinear maps  $(V'_h)^K \ni \rho \mapsto (\widehat{D}_{\rho}^{(k)} \widehat{X})(0) \in \mathbb{C}$ . These maps are indeed the derivatives of  $\widehat{X}$  in the Fréchet sense due to the analyticity of  $\widehat{X}$ : Define  $\hat{r} := r/\|r\|_1$ , then due to Proposition 3.17 and Lemma 3.13 the estimate

$$\begin{aligned}
\frac{1}{\|r\|^{K+1}} \left| \widehat{X}(r^b) - \sum_{k=0}^K \frac{1}{k!} (\widehat{D}_{(r^b, \dots, r^b)}^{(k)} \widehat{X})(0) \right| &= \frac{1}{\|r\|^{K+1}} \left| \left\langle \tau_{r^b}^*(X) - \sum_{k=0}^K \frac{1}{k!} (D_{r^b})^k X \right\rangle_0 \right| \\
&= \frac{1}{\|r\|^{K+1}} \left| \left\langle \sum_{k=K+1}^{\infty} \frac{1}{k!} (D_{r^b})^k X \right\rangle_0 \right| \\
&\leq \left| \left\langle \sum_{k=K+1}^{\infty} \frac{1}{k!} (D_{\hat{r}^b})^k X \right\rangle_0 \right| \\
&\leq \sum_{k=K+1}^{\infty} \frac{1}{k!} \|(D_{\hat{r}^b})^k X\|_1^{\bullet} \\
&\leq \sum_{k=K+1}^{\infty} \frac{1}{\sqrt{k!}} \|X\|_2^{\bullet}
\end{aligned}$$

$$\leq C \|X\|_2^\bullet$$

with  $C = \sum_{k=K+1}^\infty \frac{1}{\sqrt{k!}} < \infty$  holds uniformly for all  $r \neq 0$  with  $\|r\|_1 \leq 1$ .  $\square$

The formal deformation quantization of a Hilbert space in a very similar setting has already been examined in [6] by Dito. There the formal deformations of exponential type of a certain algebra  $\mathcal{F}_{HS}$  of smooth functions on a Hilbert space  $\mathcal{H}$  was constructed. More precisely,  $\mathcal{F}_{HS}$  consists of all smooth (in the Fréchet sense) functions  $f$  whose derivatives fulfill the additional condition that for all  $\sigma \in \mathcal{H}$

$$k! \langle f | f \rangle^k(\sigma) := \sum_{i \in I^k} |(\hat{D}_{(e_{i_1}, \dots, e_{i_k})}^{(k)} f)(\sigma)|^2 < \infty \quad (4.2)$$

holds and depends continuously on  $\sigma$  for one (hence all) Hilbert base  $e \in \mathcal{H}^I$  of  $\mathcal{H}$  indexed by a set  $I$ . In this case  $\langle f | f \rangle^k \in \mathcal{F}_{HS}$  holds.

The convergent deformations discussed in this article and the formal deformations discussed by Dito in [6] are very much analogous: In both cases it is necessary to restrict the construction to a subalgebra of all smooth functions,  $\mathcal{F}_{HS}$  or  $\mathcal{C}^{\omega_{HS}}(V'_h)$ , where the additional requirement is that all the derivatives of fixed order (in the formal case) or of all orders (in the convergent case) at every point  $\sigma$  obey a Hilbert-Schmidt condition and that the square of the corresponding Hilbert-Schmidt norms,  $\langle f | f \rangle^k(\sigma)$  or  $\langle f | f \rangle^\bullet(\sigma)$ , respectively, depend in a sufficiently nice way on  $\sigma$  such that one can prove that  $\langle f | f \rangle^k$  and  $\langle f | f \rangle^\bullet$  are again elements of  $\mathcal{F}_{HS}$  or  $\mathcal{C}^{\omega_{HS}}(V'_h)$  (see the proof of Proposition 3.4 in [6] and our Proposition 3.23). Moreover, the results concerning equivalence of the deformations are similar: In [6, Thm. 2] it is shown that two (formal) deformations are equivalent if and only if they differ by bilinear forms of Hilbert-Schmidt type, while our Theorem 3.10 shows that the corresponding equivalence transformations are continuous if and only if they are generated by bilinear forms of Hilbert-Schmidt type.

## 4.2 Deformation Quantization of Nuclear Spaces

We conclude this article with a short discussion of the case that  $V$  is nuclear. It is well known that the topology of a nuclear space can be described by continuous Hilbert seminorms. Moreover, the topology of the Hilbert tensor product on  $\mathcal{S}^k(V)$  coincides with the topology of the projective tensor product which was examined in [23]. However, for the comparison of the topologies on  $\mathcal{S}^\bullet(V)$  we have to be more careful: Let  $\|\cdot\|_\alpha \in \mathcal{P}_V$  be given. Define the seminorm  $\|\cdot\|_{\alpha, \text{pr}}^\bullet$  as

$$\|X\|_{\alpha, \text{pr}}^\bullet := |\langle X \rangle_0| + \sum_{k=1}^\infty \sqrt{k!} \inf \sum_{i \in I} \prod_{m=1}^k \|x_{i,m}\|_\alpha \quad (4.3)$$

for all  $X \in \mathcal{T}_{\text{alg}}^\bullet(V)$ , where the infimum runs over all possibilities to express  $\langle X \rangle_k$  as a finite sum of factorizing tensors, i.e. as  $\langle X \rangle_k = \sum_{i \in I} x_{i,1} \otimes \dots \otimes x_{i,k}$  with  $x_i \in V^k$ .

**Lemma 4.3** *One has the estimate*

$$\|X\|_\alpha^\bullet \leq \|X\|_{\alpha, \text{pr}}^\bullet \quad (4.4)$$

for all  $X \in \mathcal{T}_{\text{alg}}^\bullet(V)$ . Moreover, if there is a  $\|\cdot\|_\beta \in \mathcal{P}_V$ ,  $\|\cdot\|_\beta \geq \|\cdot\|_\alpha$ , such that for every  $\langle \cdot | \cdot \rangle_\beta$ -orthonormal  $e \in V^d$  and all  $d \in \mathbb{N}$  the estimate  $\sum_{i=1}^d \|e_i\|_\alpha^2 \leq 1$  holds, then

$$\|X\|_{\alpha, \text{pr}}^\bullet \leq \|X\|_\beta^\bullet \quad (4.5)$$

for all  $X \in \mathcal{T}_{\text{alg}}^\bullet(V)$ .

PROOF: Let  $X \in \mathcal{T}_{\text{alg}}^\bullet(V)$  be given, then  $\|X\|_\alpha^\bullet \leq \sum_{k=0}^\infty \|\langle X \rangle_k\|_\alpha^\bullet$  and  $\|X\|_{\alpha, \text{pr}}^\bullet = \sum_{k=0}^\infty \|\langle X \rangle_k\|_{\alpha, \text{pr}}^\bullet$ . Thus it is sufficient for the first estimate to show that  $\|\langle X \rangle_k\|_\alpha^\bullet \leq \|\langle X \rangle_k\|_{\alpha, \text{pr}}^\bullet$  for all  $k \in \mathbb{N}_0$ . Fix  $k \in \mathbb{N}_0$  and assume that  $\langle X \rangle_k = \sum_{i \in I} x_{i,1} \otimes \cdots \otimes x_{i,k}$  with  $x_i \in V^k$ . Then

$$\|\langle X \rangle_k\|_\alpha^\bullet \leq \sum_{i \in I} \|x_{i,1} \otimes \cdots \otimes x_{i,k}\|_\alpha^\bullet = \sqrt{k!} \sum_{i \in I} \prod_{m=1}^k \|x_{i,m}\|_\alpha$$

shows that  $\|\langle X \rangle_k\|_\alpha^\bullet \leq \|\langle X \rangle_k\|_{\alpha, \text{pr}}^\bullet$ , hence  $\|X\|_\alpha^\bullet \leq \|X\|_{\alpha, \text{pr}}^\bullet$ . For the second estimate, let  $\|\cdot\|_\beta$  with the stated properties and  $X \in \mathcal{T}_{\text{alg}}^k(V)$  be given. Use Lemma 2.3 to construct  $X_0 = \sum_{a \in A} x_{a,1} \otimes \cdots \otimes x_{a,k}$  and  $\tilde{X} = \sum_{a' \in \{1, \dots, d\}^k} X^{a'} e_{a'_1} \otimes \cdots \otimes e_{a'_k}$  with  $e \in V^k$  orthonormal with respect to  $\langle \cdot | \cdot \rangle_\beta$ . Clearly  $\|X_0\|_{\alpha, \text{pr}}^\bullet = 0$  and so

$$\begin{aligned} \|X\|_{\alpha, \text{pr}}^\bullet &\leq \|\tilde{X}\|_{\alpha, \text{pr}}^\bullet \\ &\leq \sqrt{k!} \sum_{a' \in \{1, \dots, d\}^k} |X^{a'}| \prod_{m=1}^k \|e_{a'_m}\|_\alpha \\ &\stackrel{\text{cs}}{\leq} \left( k! \left( \sum_{a' \in \{1, \dots, d\}^k} |X^{a'}|^2 \right) \left( \sum_{a' \in \{1, \dots, d\}^k} \prod_{m=1}^k \|e_{a'_m}\|_\alpha^2 \right) \right)^{\frac{1}{2}} \\ &\leq \left( k! \left( \sum_{a' \in \{1, \dots, d\}^k} |X^{a'}|^2 \right) \left( \sum_{i=1}^d \|e_i\|_\alpha^2 \right)^k \right)^{\frac{1}{2}} \\ &\leq \|X\|_\beta^\bullet. \end{aligned} \quad \square$$

**Proposition 4.4** *Let  $V$  be a nuclear space, then the topology on  $\mathcal{S}^\bullet(V)$  coincides with the one constructed in [23] for  $R = \frac{1}{2}$ .*

PROOF: This is a direct consequence of the preceding lemma because the locally convex topology constructed in [23] for  $R = \frac{1}{2}$  is the one defined by the seminorms  $\|\cdot\|_{\alpha, \text{pr}}^\bullet$  for all  $\|\cdot\|_\alpha \in \mathcal{P}_V$  and because in a nuclear space, such seminorms  $\|\cdot\|_\beta$  as required in the lemma exist for all  $\|\cdot\|_\alpha \in \mathcal{P}_V$ , see e.g. [15, Satz 28.4] or also [11, Chap. 21.2, Thm. 1].  $\square$

From [23, Thm. 4.10] we get:

**Corollary 4.5** *Let  $V$  be a nuclear space, then  $\mathcal{S}^\bullet(V)$  is nuclear.*

And conversely, our Theorem 3.5 implies:

**Corollary 4.6** *Let  $V$  be a nuclear space, then the  $R = \frac{1}{2}$  topology constructed in [23] is the coarsest one possible under the conditions of Theorem 3.5 in the truly (not graded) symmetric case.*

As all continuous bilinear forms on a nuclear space  $V$  are automatically of Hilbert-Schmidt type (see [11, Chap. 21.3, Thm. 5] or use [15, Satz 28.4]), we also see that the equivalence transformations  $e^{\Delta_b}$  are continuous for all continuous symmetric bilinear forms  $b$  on  $V$ , which corresponds to [23, Prop. 5.9]. Our discussion of translations and evaluation functionals then shows the existence of point-separating many positive linear functionals on the deformed algebras:

**Theorem 4.7** *Let  $V$  be a Hausdorff nuclear space and  $\overline{\cdot}$  a continuous antilinear involution of  $V$  as well as  $\Lambda$  a continuous Hermitian bilinear form on  $V$ , then there exist point-separating many continuous positive linear functionals of  $(\mathcal{S}^\bullet(V), \star_\Lambda, *)$ .*



PROOF: Choose some  $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$  and define a bilinear form  $b$  on  $V$  by  $b(v, w) := \langle \bar{v} | w \rangle_\alpha$  for all  $v, w \in V$ . Then  $b$  is continuous and Hermitian by construction and symmetric due to the compatibility of  $\langle \cdot | \cdot \rangle_\alpha$  with  $\bar{\cdot}$ . Moreover,  $\Lambda(\bar{v}, v) \leq \|\bar{v}\|_\alpha \|v\|_\alpha = \|v\|_\alpha^2 = \langle v | v \rangle_\alpha = b(\bar{v}, v)$  holds for all  $v \in V$  and  $b$  is of Hilbert-Schmidt type because every continuous bilinear form on a nuclear space is of Hilbert-Schmidt type (again, see [11, Chap. 21.3, Thm. 5] or use [15, Satz 28.4]). Because of this, Theorem 3.31 applies.  $\square$

**Remark 4.8** As Theorem 4.7 shows the existence of many continuous positive linear functionals in the nuclear case, this might be the best candidate for applications, because it allows to combine most of our results: The space  $\mathcal{S}^\bullet(V)^{\text{cpl}}$  has a clear interpretation as a space of certain analytic functions (Theorem 3.26) and its topology is essentially the coarsest possible one (Theorem 3.5). The usual equivalences of star products that are generated by continuous bilinear forms that differ only in the symmetric part still holds due to Theorem 3.10 and because all symmetric bilinear forms on a nuclear space are of Hilbert-Schmidt type. Finally, the existence of many continuous positive linear functionals assures that there exist non-trivial representations of the deformed algebras, in which all elements of up to degree 2 – which include the most important elements from the point of view of physics, e.g. the Hamiltonian of the harmonic oscillator – are represented by essentially self-adjoint operators (Theorem 3.40). Note that these results are very similar to the well-known properties that make  $C^*$ -algebras interesting for applications in physics, even though the topology on the algebra that we have considered here is far from  $C^*$ , indeed not even submultiplicative.

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